ON THE EXTENSION OF LIPSCHITZ, LIPSCHITZ-
HÖLDER CONTINUOUS, AND MONOTONE
FUNCTIONS

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1. Introduction. The well-known theorem of Kirszbraun [9], [14] asserts that a Lipschitz function from $\mathbb{R}^n$ to itself, with domain a finite point-set, can be extended to a larger domain including any arbitrarily chosen point. (The Euclidean norm is essential; see Schönbeck [16], Grünbaum [8].) This theorem was rediscovered by Valentine [17] using different methods. The writer [12] proved the same fact for a "monotone" function, and Grünbaum [9] combined these two theorems into one. A further improvement to the writer's theorem was given by Debrunner and Flor [6], who showed that the desired new functional value could always be chosen in the convex hull of the given functional values; several different proofs of this fact have now been given (see [14], [3]). An easy consequence of Kirszbraun's theorem is that a Lipschitz function in Hilbert space with maximal domain is everywhere-defined (see [11], [13]).

It was shown by S. Banach [1] that a real-valued function defined on a subset of a metric space and satisfying $|f(y_1) - f(y_2)| \leq \delta(y_1, y_2)^\alpha$, with $0 < \alpha \leq 1$ (we call this "Lipschitz-Hölder continuity"), can be extended to the whole metric space so as to satisfy the same inequality. Banach's theorem was rediscovered by Czipszer and Gehér [4] in case $\alpha = 1$ (but note that Banach's result follows, since $\delta(y_1, y_2)^\alpha$ is another metric if $\alpha \leq 1$). For a general review of the above subjects, see the article of Danzer, Grünbaum, and Klee [5]; see also [7].

In this paper, we give a unified method for proving all the above results, and also new theorems, the most striking of which is the following generalization of the Kirszbraun and Banach theorems:

THEOREM 1. Let $H$ be a Hilbert space, $M$ a metric space, $D \subset M$. Suppose $f: D \to H$ satisfies $\|f(y_1) - f(y_2)\| \leq \delta(y_1, y_2)^\alpha$ ($0 < \alpha \leq 1$). Then there exists an extension of $f$ to all of $M$ satisfying the same inequality, if either

(i) $\alpha \leq \frac{1}{2}$, or

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(ii) \( M \) is an inner product space, with metric given by \( k|\alpha|\|y_1 - y_2\| \), where \( k > 0 \).

Moreover, the extension can be performed so that the range of the extension lies in the closed convex hull of the range of \( f \); thus

\[
\|f\|_\alpha = \sup_v \|f(y)\| + \sup_{v_1, v_2} \frac{\|f(y_1) - f(y_2)\|}{\delta(y_1, y_2)^\alpha}
\]

is not increased.

(Note that in case (ii), the inequality reads \( \|f(y_1) - f(y_2)\| \leq k\|y_1 - y_2\|^\alpha \). The important point is that \( k \) need not be changed when the extension is performed.) To the best of the writer’s knowledge, no theorems on extension of Hölder-continuous functions with infinite-dimensional range have been known until now, and the present theorem is new even for finite-dimensional Hilbert space.

2. \textbf{Main theorem.} Let \( X \) be a vector space over the real numbers. A real-valued function on \( X \) is called \textit{finitely lower semicontinuous} if its restriction to any finite-dimensional subspace of \( X \) is lower semicontinuous, the subspace being taken with the “usual” topology. (Examples are: a linear function, a quadratic form; neither need be “bounded”.) Now let \( Y \) also be a space. A function \( \Phi: X \times Y \times Y \rightarrow \mathbb{R} \), written \( \Phi(x, y_1, y_2) \), shall be called a \textit{Kirszbraun function (K-function)} provided: (1°) for each fixed \( y_1, y_2 \) it is a finitely lower semicontinuous, convex function of \( x \); and (2°) for any sequence \( (x_1, y_1), \ldots, (x_m, y_m) \) in \( X \times Y \), any \( y \in Y \), and any probability vector \( (\mu_1, \ldots, \mu_m) \), we have

\[
\sum_{i=1}^{m} \mu_i \Phi(x_i - x_j, y_i, y_j) \geq 2 \sum_{i=1}^{m} \mu_i \Phi(x_i - x, y_i, y)
\]

where \( x \) stands for \( \sum_j \mu_j x_j \).

If \( X \) is a finite-dimensional space, we shall call \( \Phi \) a \textit{finite-dimensional K-function} if it satisfies the above definition with \( m \) replaced by \( 1 + \dim X \).

\textbf{Theorem 2 (Main theorem).} (A) Let \( X \) and \( Y \) be as above, and \( \Phi \) be a K-function. Let \( (x_1, y_1), \ldots, (x_m, y_m) \) be a sequence in \( X \times Y \) such that \( \Phi(x_i - x_j, y_i, y_j) \leq 0 \) for all \( i, j \), and let \( y \) be any element of \( Y \). Then there exists a vector \( x \) such that \( \Phi(x_i - x, y_i, y) \leq 0 \) for all \( i \). Furthermore, \( x \) can be chosen in the convex hull of \( \{x_1, \ldots, x_m\} \).

(B) The same statement holds if \( X \) is finite-dimensional, and \( \Phi \) is a corresponding finite-dimensional K-function.
PROOF. (A) Let $P_m$ be the set of probability-vectors in $\mathbb{R}^m$. Consider $\Phi: P_m \times P_m \to \mathbb{R}$, defined as $\Phi(\mu, \lambda) = \sum_i \mu_i \phi(x_i - x, y_i, y)$ where $x$ stands for $\sum \lambda_i x_i$. Now, $P_m$ is compact; also, $\Phi$ is convex and lower semicontinuous in $\lambda$ and concave and upper semicontinuous in $\mu$. Thus, by von Neumann’s Minimax Theorem [2] there exists a pair $(\mu^0, \lambda^0)$ in $P_m \times P_m$ such that for all $(\mu, \lambda)$ in $P_m \times P_m$

\begin{equation}
\Phi(\mu^0, \lambda) \geq \Phi(\mu, \lambda^0).
\end{equation}

By putting $\lambda=\mu^0$, we see that the left-hand side of (2.2) is nonpositive; by putting $\mu$ a Kronecker delta on the right, we have the conclusion.

(B) First apply Helly’s Theorem (see [2]) to reduce the case of general $m$ to the case $m=n+1$; then apply the proof of (A) with $m=n+1$.

3. Examples of $K$-functions. It is easily verified that the following are $K$-functions: a negative (constant) real number, a linear form in $x$, a positive semidefinite quadratic form in $x$.

For any space $Y$ and $\delta: Y \times Y \to \mathbb{R}$ such that $\delta(y_1, y_2) \geq 0$ and $\delta(y_1, y_2) \leq \delta(y_1, y_2) + \delta(y_2, y_2)$, then $(-\delta)$ is a $K$-function. In particular, $\delta$ might be a metric on $Y$.

In case $Y$ is a space with an operation “minus” satisfying $(y_1 - y_2) - (y_1 - y_2) = y_1 - y_2$ (for example, a group, with $y_1 - y_2 = y_1 y^{-1}$), and $\psi: X \times Y \to \mathbb{R}$ satisfies

\begin{equation}
\sum_{i,j} \mu_i \mu_j \psi(x_i - x_j, y_i - y_j) \geq 2 \sum_i \mu_i \psi(x_i - x, y_i)
\end{equation}

then $\Phi(x, y_1, y_2) = \psi(x, y_1 - y_2)$ satisfies the inequality of the definition of “$K$-function.” If $Y$ is a linear space, then $\psi$ might be a negative semidefinite quadratic form in $y$, or a bilinear form in $x$ and $y$; these give rise to $K$-functions.

If $x$ is the real numbers, then $x^4$ is a $K$-function; this follows from the identity

$$
\sum \mu_i \mu_j |x_i - x_j|^4 = 2 \sum_i \mu_i |x_i - x|^4 + 6 \left( \sum_i \mu_i x_i^2 - x^2 \right)^2
$$

(where $x$ is $\sum \mu_i x_i$, as before, and $\sum \mu_i = 1$).

Moreover, any linear combination of $K$-functions with nonnegative coefficients is a $K$-function. (Of course, assuming $X$, $Y$ the same for all of them.)
Corollaries to Theorem 1. Kirszbraun’s Theorem follows from the case \( \psi(x, y) = \|x\|^2 - \|y\|^2 \). The Debrunner-Flor Lemma mentioned in the Introduction is the case where \( \psi(x, y) \) is a bilinear form. The theorem of Grünbaum [9] is contained in the case \( \psi = k_1(\|x\|^2 - \|y\|^2) + k_2(x, y) \), with nonnegative \( k_1, k_2 \).

Letting \( X \) be a Hilbert space and \( Y \) a metric space, and taking \( \Phi(x, y_1, y_2) = \|x\|^2 - \delta(y_1, y_2) \), we obtain the necessary lemma to prove part (ii) of Theorem 1, with \( \alpha = \frac{1}{2} \). The proof parallels closely the usual proof of the extension theorem for Lipschitz functions (see [11] or [13]), slightly modified to keep the range of the extension in the closed convex hull of the range of \( f \).

As remarked in the Introduction, \( [\delta(y_1, y_2)]^\beta \) is also a metric if \( \beta \leq 1 \); hence we have an extension theorem for \( f \) satisfying \( \|f(y_1) - f(y_2)\| \leq [\delta(y_1, y_2)]^\alpha \) with \( \alpha \leq \frac{1}{2} \). Indeed, if \( g(\gamma) \) is a real-valued function of \( \gamma \geq 0 \) with \( g(0) = 0 \), \( g(\gamma) > 0 \) for \( \gamma > 0 \), \( g \) nondecreasing in \( \gamma \), and \( \gamma^{-1}g(\gamma) \) nonincreasing for \( \gamma > 0 \), we have (for \( \gamma_1, \gamma_2 > 0 \)):

\[
\gamma_1 g(\gamma_1 + \gamma_2) \leq (\gamma_1 + \gamma_2) g(\gamma_1),
\]

\[
\gamma_2 g(\gamma_1 + \gamma_2) \leq (\gamma_1 + \gamma_2) g(\gamma_2)
\]

whence (by adding) \( g \) is subadditive, so that \( g \circ \delta \) is again a metric. Thus \( g(\gamma) = \gamma^\beta \), with \( \alpha \leq 1 \), is a special case.

It has recently been established by H. Brézis and C. M. Fox that \( \psi(x, y) = -\|y\|^\beta \) is a \( K \)-function for \( 0 < \beta \leq 2 \) in a Euclidean space (or an inner product space). Brézis uses M. Riesz’ Convexity Theorem; Fox gives an elementary (but ingenious) proof of the stronger statement

\[
(3.2) \quad \sum_{i,j} \mu_i \mu_j \|y_i - y_j\|^{2\alpha} \leq \sum_{i,j} \mu_i \mu_j (\|y_i\|^2 + \|y_j\|^2)^\alpha \quad \text{for } 0 < \alpha \leq 1.
\]

J. Moser and the writer have simplified Fox’s proof, as follows:

**Lemma.** For \( x_1, \cdots, x_m \) in an inner product space, and \( a_1, \cdots, a_m > 0, \beta > 0, \) note

\[
(3.3) \quad \sum_{i,j} \frac{\langle x_i, x_j \rangle}{(a_i + a_j)\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty \left\| \sum_i e^{-\alpha t} x_i \right\|^{2\beta - 1} dt
\]

and thus it is nonnegative.

Now write the left-hand side of (3.2) as

\[
\sum_{i,j} \mu_i \mu_j (\|y_i\|^2 + \|y_j\|^2)^\alpha \left[ 1 - \frac{2\langle y_i, y_j \rangle}{\|y_i\|^2 + \|y_j\|^2} \right]^\alpha,
\]
apply Bernoulli’s inequality to the expression in square brackets, and then the lemma, with $x_i = \mu_1 y_i$, and $a_i = \|y_i\|^2$. (The case where some $y_i$ are zero is easily disposed of by a continuity argument.)

The above argument is easily generalized to show $-\left[Q(y_1 - y_2)\right]^{\alpha}$, with $0 < \alpha \leq 1$, is a $K$-function if $Q$ is a positive semidefinite quadratic form in a linear space $Y$. Part (ii) of Theorem 1 is proved by use of the $K$-function $\|x\|^2 - k^2 sym\|y_1 - y_2\|^{2\alpha}$, followed by the “usual” argument for Lipschitz functions.

J. Moser and G. Schober have shown that if $X$ is one-dimensional, then $-\left[\delta(y_1, y_2)\right]^2$ is a finite-dimensional $K$-function; i.e., it satisfies the desired inequality with $m = 2$. Schober’s proof considers separately the case $\delta(y_1, y_2)^2 \leq \delta(y_1, y)^2 + \delta(y_2, y)^2$ which is easy, and the opposite case, which is treated by the standard maximization argument of differential calculus applied to the function $f(\mu) = \mu(1 - \mu)\delta(y_1, y_2)^2 - \mu\delta(y_1, y)^2 - (1 - \mu)\delta(y_2, y)^2$. The extension theorem of Banach follows by Theorem 2, part (B), applied to $|x|^2 - [\delta(y_1, y_2)]^2$.

NOTE ADDED IN PROOF. Banach’s theorem mentioned above is more probably due to McShane (Bull. Amer. Math. Soc. 40 (1934), 837-842). (2°) The hypothesis “finitely lower-semicontinuous” follows from the other hypotheses of the definition of “$K$-function”, and so can be dropped. (3°) Hayden, Wells, and Williams of the University of Kentucky have generalized the extension-theorem to cover functions from one $L^p$-space to another (unpublished work).

REFERENCES


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