The theorem of A. Gleason [2, vii.23] asserts that every continuous map $f$ from an open subset $U$ of a product $X$ of separable topological spaces into a Hausdorff space $Y$ whose points are $G_\delta$-sets has the form $g \circ \pi|_U$, where $\pi$ is a countable projection of $X$ and $g : \pi(U) \to Y$ is continuous. A natural question is to find what other "pleasant" subsets $U$ of $X$ have the above factorization property. The most plausible ones are compact subsets: for, if $U \subseteq X$ is compact and $f = g \circ \pi|_U$ with $f$ continuous, then $g$ must be continuous since $\pi|_U$ is a closed map (being continuous on a compact space).

The first part of this note rejects this conjecture by giving an example of a compact subset of a product of copies of the unit interval, without the factorization property. In the second part, it is proved that the factorization $f = g \circ \pi|_U$ always holds whenever $f$ is uniformly continuous and the range metric. This result implies an open mapping theorem for continuous linear mappings on products of Fréchet spaces.

1. The example. Let $Z$ be a compact Hausdorff space which is first countable but not metrizable. Such a space exists by [1, §2, Exercise 13]. Since $Z$ is completely regular, $Z$ is homeomorphic to a compact subset $U$ of a product $X$ of copies of $[0, 1]$. Let $f : U \to U$ be the identity. Assume that $f = g \circ \pi|_U$, with $\pi$ a countable projection and $g : \pi(U) \to U$ continuous, and argue for a contradiction. Since countable products of separable metric spaces are separable metric, $\pi(U)$ is separable metric. Hence $U$ is a continuous image of a separable metric space. But a cosmic metric space is metrizable whenever it is compact by [3, p. 994, (C) for cosmic spaces]. This contradicts the assumptions on $Z$.

2. A factorization theorem. The above example shows that the following result does not hold longer when $Y$ is not metrizable.

**Theorem.** If $Z$ is any subset of a product of arbitrary uniform spaces

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A. GLEASON FACTORIZATION THEOREM

$X_\alpha (\alpha \in A)$ into a metric space $Y$, then every uniformly continuous $f: Z \to Y$ has the form $g \circ \pi |_Z$ with $\pi$ a countable projection and $g$ uniformly continuous.

**Proof.** By the uniform continuity of $f$, for each integer $n \geq 1$ there are a finite subset $A_n \subseteq A$ and uniform covers $\mathcal{U}_n$ of $X_\alpha (\alpha \in A_n)$ such that

$$d(f(x), f(y)) \leq 1/n$$

whenever $x, y \in Z$ have the coordinates corresponding to $\alpha \in A_n$ near of order $\mathcal{U}_n$. Put $C = \bigcup_{n=1}^{\infty} A_n$ and $\pi$ the countable projection $(x_\alpha)_{\alpha \in A} \to (x_\alpha)_{\alpha \in C}$. For every $x \in \pi(Z)$, let $z_x$ be a point of $Z \setminus \pi^{-1}(x)$. Define $g: \pi(Z) \to Y$ by $x \to f(z_x)$. If $z', z'' \in Z$ have the same image by $\pi$, then $d(f(z'), f(z'')) \leq 1/n$ for all $n \geq 1$ (since $C \supseteq A_n$), which implies $d(f(z'), f(z'')) = 0$, i.e. $f(z') = f(z'')$. This means that $g$ is well defined. From the definition it follows $f = g \circ \pi |_Z$. The equality $f = g \circ \pi |_Z$ means that two points of $Z$ have the same image by $f$ whenever they have the same coordinates for $\alpha \in C$. By this and $C \supseteq A_n (n \geq 1)$, $g$ is uniformly continuous. Q.E.D.

**Corollary.** Let $X_\alpha (\alpha \in A)$, $Y$ be arbitrary complete metrizable topological vector spaces. Then every continuous linear map $f$ from $\prod_{\alpha \in A} X_\alpha$ onto $Y$ is open.

**Proof.** Since a continuous linear map is uniformly continuous in the standard uniformities of topological vector spaces, the above theorem implies that $f = g \circ \pi$, with $\pi$ a countable projection and $g$ uniformly continuous. Since $\pi$ and $f$ are linear, $g$ is linear. Since a countable product of complete metric spaces is complete metric, $g$ is open by Banach homomorphism theorem. Since $\pi$ is open, $f$ must be also. Q.E.D.

**References**


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