A CLASSIFICATION OF MODULES OVER COMPLETE DISCRETE VALUATION RINGS

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1. Introduction. The purpose of this paper is to announce the completion of a classification (up to isomorphism) of all modules which are direct sums of countably generated modules over complete discrete valuation rings. The detailed proofs will appear elsewhere. Throughout this paper, let \( R \) denote a fixed but arbitrary complete discrete valuation ring and \( p \) a fixed but arbitrary prime element of \( R \). For the sake of convenience, a cardinal is viewed as the first ordinal having that cardinality. Let \((c, R, k)\) be the class of all countably generated reduced \( R \)-modules of (torsion-free) rank \( \leq k \) and \( D(c, R, k) \) that of all direct sums of members of \((c, R, k)\). Clearly

\[
(c, R, 0) \subset (c, R, 1) \subset \cdots \subset (c, R, \omega)
\]

\[
D(c, R, 0) \subset D(c, R, 1) \subset \cdots \subset D(c, R, \omega).
\]

Notice that a \( p \)-primary abelian group is a member of \((c, R, 0)\), particularly if \( R \) is a ring of \( p \)-adic integers. A classification (of all members) of \((c, R, k)\) was done by Ulm (1933) when \( k = 0 \) [8], by Kaplansky and Mackey (1951) when \( k = 1 \) [4], by Rotman and Yen (1961) when \( k < \omega \) [7], and that of \( D(c, R, k) \) was done by Kolettis (1960) when \( k = 0 \) [5]. First, we complete a classification of \((c, R, \omega)\) and then, utilizing this, we finish that of \( D(c, R, \omega) \).

2. Invariants. We need two kinds of invariants, namely, the Ulm invariants and the basis types. Since the celebrated Ulm invariants are well known, a brief explanation of the basis types only is in order [2], [4], [7]. Let \( R^k = \oplus \{ R : i < k \} \) for each \( k \). Define \( f(R) \) to be the class of all sordinal (ordinal or \( \omega \)) valued functions on \( R^k \) for all cardinals \( k \), and \( m(Q) \) that of all square row-finite matrices over \( Q \), the quotient field of \( R \). Suppose that \( f, g \subseteq f(R) \). Define \( f \sim g \) to mean

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both that $\text{Dom } f = \text{Dom } g = R^k$ for some cardinal $k$ and that there is a matrix $\gamma$ and a diagonal matrix $\delta$, both $k \times k$ invertible integral (that is, all entries are elements of $R$) in $m(Q)$, such that $f(\alpha \gamma) = g(\alpha \delta)$ for all $\alpha \in R^k$. It is routine to show that $\sim$ is an equivalence relation on $f(R)$.

Let $M$ be an $R$-module of rank $k$. Then, every basis $\eta = \{y_i: i < k\}$ defines a function $g$ of $f(R)$ by

$$g(\alpha) = h_p(\alpha \eta) = h_p(\sum a_i y_i: i < k)$$

for all $\alpha = \{a_i: i < k\} \in R^k$. Notice that $g = \infty$ if $k = 0$ since a sum without term is 0. It is routine to show that $g \sim g'$ if $g'$ is defined by another basis of $M$. Thus, $M$ determines uniquely a class of $f(R)/\sim$, which we call the basis type of $M$. It is easy to show the following lemma.

**Lemma 1.** Two reduced $R$-modules $M$ and $M'$ have the same basis type if and only if they contain basic free submodules $F$ and $F'$, respectively, with a height-preserving isomorphism from $F$ onto $F'$.

3. A classification of $(c, R, \omega)$.

**Theorem 1.** Let $M$ and $M'$ be countably generated reduced $R$-modules. Then, $M \sim M'$ if and only if they have the same Ulm invariant and the same basis type.

Only the "if" part needs a proof. Let $\alpha = \{a_i: i < k\}$. Define $\alpha(r) = \{a_i: i < r\}$ for each number $r$. Let $k$ be the same rank of $M$ and $M'$. Then, by Lemma 1, there are ordered bases $\eta = \{y_i: i < k\}$ and $\eta' = \{y'_i: i < k\}$ of $M$ and $M'$, respectively, with a height-preserving isomorphism $\rho$ such that $\rho(\alpha \eta) = \alpha \eta'$ for all $\alpha \in R^k$. We may assume that there are countable subsets $\xi = \{x_i: i < \omega\}$ and $\xi' = \{x'_i: i < \omega\}$ of $M$ and $M'$, respectively, such that

$$M = [\xi \cup \eta] \quad \text{and} \quad M' = [\xi' \cup \eta'] \quad \text{with}$$

$$px_i \in [\xi(i) \cup \eta(i)] \quad \text{and} \quad px'_i \in [\xi'(i) \cup \eta'(i)] \quad \text{for each } i < \omega.$$ 

The main idea of the proof is to construct a sequence of height-preserving isomorphisms $\{\phi_i: i < \omega\}$ in such a way that the following conditions are satisfied.

(a) $\phi_i: A_i \rightarrowtail A'_i$ where

$$A_i = [\xi(i) \cup \eta(i) \cup \phi_i^{-1}(\xi'(i) \cup \eta'(i))],$$

$$A'_i = [\xi'(i) \cup \eta'(i) \cup \phi_i(\xi(i) \cup \eta(i))].$$

(b) $\phi_0 \leq \cdots \leq \phi_i \leq \phi_{i+1} \leq \cdots$.

(c) There exists a nonnegative integer $n(i)$ such that $p^{n(i)} A_i$
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\[ \phi_i \subseteq [\eta(i)] \text{ and } \rho^n(i) A_i = [\eta'(i)] \text{ and } \phi_i = \rho \text{ as height-preserving isomorphism from } \rho^n(i) A_i \text{ onto } \rho^n(i) A_i'. \]

The supremum of \( \{ \phi_i : i < \omega \} \) gives the required isomorphism from \( M \) onto \( M' \). For more detailed proof, see [1] or [2].

4. A classification of \( D(c, R, \omega) \).

**Theorem 2.** Let \( M \) and \( M' \) be direct sums of countably generated reduced \( R \)-modules. Then, \( M \cong M' \) if and only if they have the same Ulm invariant and the same basis type.

Again, only the “if” part needs a proof. We may write as

\[ M = \bigoplus \{ M_i : i \in I \} \quad \text{and} \quad M' = \bigoplus \{ M'_i : i \in I \} \]

where all \( M_i, M'_i \in (c, R, \omega) \) and \( I \) is a cardinal. For notational convenience, define \( M(T) = \bigoplus \{ M_i : i \in T \} \), \( T \subseteq I \). The main idea of the proof is to show that there is a partition of \( I \) into countable subsets \( \{ I_j : j < I \} \) such that, for each \( j < I \), \( M(I_j) \) and \( M'(I_j) \) have the same Ulm invariant and the same basis type. Then by Theorem 1, they are isomorphic and, consequently, \( M \cong M' \). In fact, by the Kolettis theorem [3], [5], [6], we may assume that \( M_i \) and \( M'_i \) have already the same Ulm invariant for each \( i \). The following lemmas indicate the route of the proof.

**Lemma 2.** Let \( N = A \oplus B \oplus C \) be a reduced \( R \)-module such that the following conditions are satisfied.

(a) There are in \( N \) disjoint subsets \( \eta_A \) and \( \eta_B \) such that \( \eta_A \) and \( \eta_A \cup \eta_B \) are bases of \( A \) and \( A \oplus B \), respectively.

(b) If \( x_A \in [\eta_A] \) and \( x_B \in [\eta_B] \), then \( (h_N \text{ denotes the } p\text{-height in } N) \)

\[ h_N(x_A + x_B) = \min\{ h_N(x_A), h_N(x_B) \}. \]

Then, if we write \( \eta_B = \{ y_i : i < k \} \), \( k = |\eta_B| \), there is in \( m(Q) \) a \( k \times k \) diagonal invertible integral matrix \( \delta = \{ d_i : i < k \} \) such that the following conditions are satisfied.

(c) \( \tau = \Pi_B(\delta \eta_B) \) is an ordered basis of \( B \). (Here, \( \Pi_B \) is the canonical projection of \( N \) onto \( B \).)

(d) There is a height-preserving isomorphism \( \rho \) from \([\delta \eta_B]\) onto \([\tau]\) such that \( \rho(\alpha \delta \eta_B) = \alpha \tau \) for all \( \alpha \in R^k \).

**Lemma 3.** Let \( k \) be the rank of \( M \) and \( M' \). Let \( \eta = \{ y_i : i < k \} \) and \( \eta' = \{ y'_i : i < k \} \) be ordered bases of \( M \) and \( M' \), respectively, with \( \eta' \) summandwise (that is, each \( y'_i \in M_j \) for a \( j \)). If \( J \) is a countable subset of \( I \), then there is a set \( T \) such that the following conditions are satisfied.

(a) \( T \) is countable and \( J \subseteq T \subseteq I \).
(b) Define $\eta(T) = \{ y_i \in \eta : y_i \in M(T) \}$. $\eta(T)$ and $\eta'(T)$ are bases of $M(T)$ and $M'(T)$, respectively.

(c) $y_i \in \eta(T)$ if and only if $y_i \in \eta'(T)$.

**Lemma 4.** $M$ and $M'$ have the same basis type if and only if there is a partition of $I$ into countable subsets $\{ I_j : j < I \}$ such that $M(I_j)$ and $M'(I_j)$ have the same basis type for each index $j < I$.

Using Lemma 2, 3 and a transfinite induction, we can prove Lemma 4. Theorem 2 is immediate from Lemma 4.

**Corollary.** $M \cong M'$ if and only if they have isomorphic torsion parts and contain basic free submodules $F$ and $F'$ of, respectively, with a height-preserving isomorphism from $F$ onto $F'$.

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**Bibliography**


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