REMARKS CONCERNING $\text{Ext}^\ast (M, -)$

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Let $X$ be a topological space and let $S$ (respectively $\alpha$) be the category of sets (respectively abelian groups). Let $S'$ (respectively $\alpha'$) be the category of sheaves of sets (respectively abelian groups) based on $X$, and fix a sheaf $M$ in $\alpha'$. The graded functor $\text{Ext}^\ast(M, -) : \alpha' \to \alpha$ is computed as the right derived functors of $\text{Hom}(M', -)$, and of course $\text{Ext}^i(M, N)$ classifies $i$-fold extensions of $M$ by $N$ [6].

One would also like to be able to classify extensions in nonabelian categories of sheaves. Partial success in this direction has been achieved by Gray [5], but he needs to assume restrictions on $X$ as well as on $M$. In [10], the author applied triple-theoretic [1] techniques to the category of sheaves of $R$-algebras ($R$ a sheaf of rings), and successfully classified cohomologically singular extensions of an $R$-algebra $P$ by one of its modules $N$.

Specifically, if $G$ is the polynomial algebra cotriple lifted to the category of sheaves of $R$-algebras, if $T$ is the Godement triple = standard construction [3], and if $\text{Der}_R(P, N)$ is the abelian group of global $R$-derivations from $P$ to $N$, then the equivalence classes of singular extensions of $P$ by $N$ are in one-one correspondence with the elements of the first homology group of the double complex $\text{Der}_R(G^*P, T^*N)$. In §11 of this note we prove that if $G$ is the free abelian group cotriple lifted to $\alpha'$ then the $n$th homology group of the double complex $\text{Hom}(G^*M, T^*N)$ is naturally isomorphic to $\text{Ext}^n(M, N)$. The combination of this theorem and the results in [10] indicates a unified approach to the cohomological classification of extensions in many (algebraic) categories of sheaves.

In §1 one can find a theorem which is part of the folklore of triple-theoretic cohomology theory, but for which no straightforward proof appears in print. The theorem is: if an abelian category has an injective cogenerator and $E$ is the model-induced triple then $\text{Ext}^\ast(M, N)$ and the homology of the complex $\text{Hom}(M, E^*N)$ are naturally isomorphic (note that $E$ is not the triple used by Schafer in [8]).

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The essence of the proof of this theorem appears as a proposition in §I, and we use the proposition again in §II.

I. Ext*(M, −) is a triple-derived functor. In a number of places (e.g. [7], [4], [6]) one can find a proof of the fact that the category \( \mathcal{A}' \) has an injective cogenerator. Thus one can always find an injective resolution \( J^* \) for any sheaf \( N \) in \( \mathcal{A}' \), and \( \text{Ext}^n(M, N) \) is defined to be the \( n \)th homology group of the complex \( \text{Hom}(M, J^*) \).

On the other hand, if \( I \) is the injective cogenerator then we can define the “model-induced” triple \( E = (E, \eta, \mu) \) as follows. The functor \( E: \mathcal{A}' \to \mathcal{A}' \) is given by \( EN = \prod I \) where the product is taken over the set \( \text{Hom}(N, I) \). If we write \( \langle g \rangle: \prod I \to I \) for the coordinate projection corresponding to the map \( g \) in \( \text{Hom}(N, I) \) then the natural transformations \( \eta, \mu \) are given by \( \langle g \rangle \cdot \eta N = g \) and \( \langle g \rangle \cdot \mu N = \langle \langle g \rangle \rangle \). Then \( (E, \eta, \mu) \) is a triple (see [1]). Moreover, since the product of injectives is injective, \( EN \) is injective for each \( N \). We have the complex

\[
N \to EN \to E^2N \to E^3N \to \cdots = E^*N
\]

where \( d: E^kN \to E^{k+1}N \) is \( d = \sum_{i=0}^k (-1)^i E^i \eta E^{k-i} N \), hence we can consider the homology of the complex \( \text{Hom}(M, EN) \to \text{Hom}(M, E^2N) \to \text{Hom}(M, E^3N) \to \cdots \). Denote the \( n \)th homology group of this complex by \( H^n(M, N)E \). Then we claim that \( H^n(M, N)E = \text{Ext}^n(M, N) \) for all \( n \geq 0 \).

**Lemma.** The map \( \eta N: N \to EN \) is a monomorphism.

**Proof.** If \( f, f': N \to N \) are such that \( \eta N \cdot f = \eta N \cdot f' \) then we must show that \( f = f' \). Now for each \( g: N \to I \) we have \( g \cdot f = \langle g \rangle \cdot \eta N \cdot f = \langle g \rangle \cdot \eta N \cdot f' = g \cdot f' \), and since \( I \) is a cogenerator, \( f = f' \).

The dual of the following proposition was shown to me by Michael Barr.

**Proposition.** If the abelian category \( \mathfrak{B} \) is endowed with a triple \( E \) such that \( \eta \) is pointwise monic then \( N \to E^*N \) is an exact sequence, and conversely.

**Proof.** The converse is obvious. On the other hand, if \( \eta N \) is monic for each \( N \) in \( \mathfrak{B} \) then we can build an exact sequence

\[
0 \to N \to EN \to EC_0 \to EC_1 \to EC_2 \to \cdots = I^*
\]

where \( C_{-1} = N \) and \( C_{i+1} \) is the cokernel of the map \( \eta C_i: C_i \to EC_i \) for each \( i \geq -1 \). Of course the boundary \( EC_i \to EC_{i+1} \) is the composition of the cokernel map \( c_{i+1}: EC_i \to C_{i+1} \) and \( \eta C_{i+1} \). If \( \mathfrak{E} \) is the injective
class determined by the image of $E$ then any two $\mathcal{E}$-injective and $\mathcal{E}$-exact sequences are homotopic \cite{2}. Now $N \to E^*N$ is $\mathcal{E}$-injective and $\mathcal{E}$-exact. Moreover $N \to I^*$ is $\mathcal{E}$-injective, and we now show that it is also $\mathcal{E}$-exact, i.e. that for any $N'$ we have $\text{Hom}(I^*, EN')$ is exact. Given a cocycle $f : EC_j \to EN'$ we have $0 = df = f \cdot \eta C_j \cdot c_j$ and $c_j$ is epic, hence $f \cdot \eta C_j = 0$. But $c_{j+1}$ is the cokernel of $\eta C_j$ and so there is a map $f' : C_{j+1} \to EN'$ such that $f' \cdot c_{j+1} = f$. Now the coboundary of $\mu N' \cdot Ef' : EC_{j+1} \to EN'$ is

$$d(\mu N' \cdot Ef') = \mu N' \cdot Ef' \cdot \eta C_{j+1} \cdot c_{j+1}$$
$$= \mu N' \cdot \eta EN' \cdot f' \cdot c_{j+1}$$
$$= f' \cdot c_{j+1}$$
$$= f.$$

Thus every cocycle is a coboundary, $\text{Hom}(I^*, EN')$ is exact, and $I^*$ is $\mathcal{E}$-exact. It follows that $I^*$ and $E^*N$ are homotopic. But $I^*$ is exact, hence so is $E^*N$.

**Corollary.** If $E$ is the model-induced triple on $\mathcal{A}'$ defined above then for each $N$ in $\mathcal{A}'$, $N \to E^*N$ is an injective resolution.

**Corollary.** $H^*(M, N)E \approx \text{Ext}^*(M, N)$.

**Remark.** The proof works for any abelian category having an injective cogenerator. Dually, if an abelian category has a projective generator and $P$ is the model-induced cotriple then $H^*(M, N)P \approx \text{Ext}^*(M, N)$.

**II. A double complex yielding $\text{Ext}^*(M, N)$**. Consider the following diagram of categories and functors:

\[\begin{array}{c}
\text{S} & \xrightarrow{\alpha'} & \prod \alpha \\
\text{F} \downarrow & & \downarrow \prod F_s \\
\text{U} & \xrightarrow{Q} & \prod U_s \\
\text{s'} & \xrightarrow{Q} & \prod s
\end{array}\]

in which the products are taken over all points $x$ in $X$. $S$ is the stalk functor, i.e. $S$ takes a sheaf to the set of its stalks. $Q$ takes a collection $\{A_x\}$ to the sheaf whose value at an open set $V$ is $\prod A_x$, the product being taken over all points $x$ in $V$. $U$ and $\prod U_x$ are the obvious "underlying" or "forgetful" functors. $F_s$ is the free abelian group functor. Given a sheaf $N$ in $s'$, the functor which takes an open set $V$ to the free abelian group on the set $N(V)$ is a presheaf of abelian groups, and $FN$ is defined to be the sheaf associated to this presheaf.
One can show that $S$ is left adjoint to $Q$, $F$ is left adjoint to $U$, $\prod U_x$ is left adjoint to $\prod U_x S$, and $Q \prod U_x \approx UQ$. Moreover $QS$ is the Godement “standard construction” (see [3]). Let $T = (T, \eta, \mu)$ be the triple associated to $QS = T$ and $G = (G, \epsilon, \delta)$ the cotriple associated to $FU = G$ via the adjointnesses. For each $M$ in $\mathcal{U}$ we get the complex
\[ \cdots \rightarrow G^2M \rightarrow G^2M \rightarrow GM \rightarrow M \rightarrow 0 \]
dually to the way we got $N \rightarrow E*N$ in §1. For each $N$ in $\mathcal{U}$ we get the complex $0 \rightarrow N \rightarrow TN \rightarrow T^{1}N \rightarrow \cdots$ as in §1. Hence we have the double complex
\[ C^{ij}(M, N) = \text{Hom} (G^{i+1}M, T^{j+1}N) \quad \text{for } i, j \geq 0 \]
with boundaries induced by the boundaries in the single complexes. Denote the $n$th homology group of this double complex by $H^n(M, N)_{G, T}$.

**Theorem.** $H^n(M, N)_{G, T} = \text{Ext}^n(M, N)$ for each $n \geq 0$.

**Proof.** It is well known (see [6] or [9]) that $\text{Ext}^*(M, -)$ is a cohomological $\delta$-functor augmented over $\text{Hom}(M, -)$, and that any two such cohomological $\delta$-functors are isomorphic. We thus verify that $H^*(M, -)_{G, T}$ is such a functor. For convenience we write $H^*(M, -)$ instead of $H^*(M, -)_{G, T}$.

Given an exact sequence $0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$ in $\mathcal{U}$ we need to produce an exact triangle
\[ H^*(M, N') \rightarrow H^*(M, N) \rightarrow H^*(M, N'') \rightarrow H^*(M, N'). \]
Now $T$ is an exact functor [3] and $\text{Hom}(G^{i+1}M, -)$ is left exact for $i \geq 0$. Thus
\[ 0 \rightarrow \text{Hom} (G^{i+1}M, T^{j+1}N') \rightarrow \text{Hom} (G^{i+1}M, T^{j+1}N) \rightarrow \text{Hom} (G^{i+1}M, T^{j+1}N'') \]
is exact for each $i, j \geq 0$. Moreover, the last map is onto, for consider the chain of natural isomorphisms:
\[ \text{Hom} (G^{i+1}M, T^{j+1}N) \approx \text{Hom} (UG^iM, UT^{j+1}N) \approx \text{Hom} (UG^iM, Q \prod U_x (SQ)^jSN) \approx \text{Hom} (SUG^iM, \prod U_x (SQ)^jSN) \approx \text{Hom} (SUG^iM, (SQ)^j \prod U_x SN). \]
Since $N \rightarrow N''$ is epic, so is $SN \rightarrow SN''$ (see [3]) and thus $\prod U_x SN \rightarrow \prod U_x SN''$ is a split epimorphism. Hence
\[ \text{Hom}(SUG^iM, (SQ)_j \prod U \omega SN) \rightarrow \text{Hom}(SUG^iM, (SQ)_j \prod U \omega SN") \]

is onto. But this map is naturally isomorphic to
\[ \text{Hom}(G^{i+1}M, T^{i+1}N) \rightarrow \text{Hom}(G^{i+1}M, T^{i+1}N") \]

which is therefore onto.

It follows that \( 0 \rightarrow C^{i+j}(M, N') \rightarrow C^{i+j}(M, N) \rightarrow C^{i+j}(M, N") \rightarrow 0 \) is exact for each \( i, j \geq 0 \) and that \( 0 \rightarrow C^{**}(M, N') \rightarrow C^{**}(M, N) \rightarrow C^{**}(M, N") \rightarrow 0 \) is an exact sequence of double complexes. The exact homology triangle is now a standard result of homological algebra. Hence \( H^*(M, -) \) is an exact \( \delta \)-functor.

The proof is completed by showing that \( H^*(M, -) \) is augmented over \( \text{Hom}(M, -) \) and that \( H^n(M, -) \) is effaceable for each \( n > 0 \). First, \( H^0(M, N) \) is the intersection of the kernels of the two maps \( C^0_0(M, N) \rightarrow C^0_1(M, N) \) and \( C^0_0(M, N) \rightarrow C^1_0(M, N) \). Now \( \epsilon M: GM \rightarrow M \) is an epimorphism (essentially because the associated sheaf functor is exact) and \( \eta N: N \rightarrow TN \) is a monomorphism \([3]\]. Hence by the proposition in §1 and its dual, \( N \rightarrow T^*N \) and \( G^*M \rightarrow M \) are exact sequences. But \( \text{Hom}(-, -) \) is left exact and thus
\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 \rightarrow \text{Hom}(M, N) & \rightarrow \text{Hom}(GM, N) & \rightarrow \text{Hom}(G^2M, N) \\
\downarrow & \downarrow & \downarrow \\
0 \rightarrow \text{Hom}(M, TN) & \rightarrow \text{Hom}(GM, TN) & \rightarrow \text{Hom}(G^2M, TN) \\
\downarrow & \downarrow & \downarrow \\
0 \rightarrow \text{Hom}(M, T^2N) & \rightarrow \text{Hom}(GM, T^2N) & \\
\end{array}
\]

is exact. This implies that \( H^0(M, N) \approx \text{Hom}(M, N) \) and \( H^*(M, -) \) is augmented over \( \text{Hom}(M, -) \).

Finally, to demonstrate the effaceability, let \( N \) be injective in \( \mathcal{A}' \). Then \( \eta N: N \rightarrow TN \) is a split monomorphism, say \( u \cdot \eta N = N \). As is shown in \([1]\], the maps \( T^j u \) provide a contraction of the complex \( 0 \rightarrow N \rightarrow T^*N \). Thus for each \( j \geq -1 \) the column \( C^j(M, N) \) is exact and has zero homology. It follows that the total homology of \( C^{**}(M, N) \) vanishes in positive dimensions, that is, \( H^n(M, N) = 0 \) if \( n > 0 \) and if \( N \) is injective. Hence \( H^n(M, -) \) is effaceable for \( n > 0 \). This completes the proof of the fact that \( H^*(M, -) \) is a cohomological \( \delta \)-functor augmented over \( \text{Hom}(M, -) \), which was to be shown.

References


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