AN INVARIANCE PRINCIPLE FOR THE EMPIRICAL PROCESS WITH RANDOM SAMPLE SIZE

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Let \( C = C[0, 1] \) be the space of continuous functions on \([0, 1]\) with the uniform topology, that is the distance between two points \( x \) and \( y \) (two functions \( x \) and \( y \) of \( t \in [0, 1] \)) is defined by

\[
\rho(x, y) = \sup_t |x(t) - y(t)|.
\]

Let \( \mathfrak{B} \) be the \( \sigma \)-field of Borel sets of \( C \).

Let \((\Omega, \mathcal{A}, P)\) be some probability space and \( W \) be the Wiener measure on \((C, \mathfrak{B})\) with the corresponding Wiener process \( \{W_t(\omega) : 0 \leq t \leq 1\} \), \( \omega \in \Omega \); that is \( W_t \) has values in \( C \) and is specified by \( E(W_t) = 0 \) and \( E(W_s W_t) = s \) if \( s \leq t \). Let \( W^0 \) be the Gaussian measure on \((C, \mathfrak{B})\) constructed by setting \( W^0_t = W_t - t W_1 \). Then \( W^0_t \in C \), \( E(W^0_t) = 0 \) and \( E(W^0_s W^0_t) = s(1-t) \) if \( s \leq t \). Also \( W^0_0 = W^0_1 = 0 \) with probability 1 and \( \{W^0_t : 0 \leq t \leq 1\} \) is called the tied down Wiener process or the Brownian bridge.

Let \( \xi_n = \xi_1 + \cdots + \xi_n \), \( S_0 = 0 \), \( n = 1, 2, \cdots \) be the partial sum sequence of random variables \( \{\xi_n\} \) defined on \((\Omega, \mathcal{A}, P)\). Define a random element \( X_n \) of \( C \) by

\[
(1) \quad X_n(t, \omega) = W_n(t, \omega) + (nt - [nt])\xi_{[nt]+1}(\omega)/n^{1/2} - t W_n(1, \omega)
\]

where \( W_n(t, \omega) = S_{[nt]}(\omega)/n^{1/2} \). The following theorem is an immediate consequence of L. Breiman’s analysis of §§13.5 and 13.6 in his book [3].

**Theorem B.** Suppose the random variables \( \xi_1, \xi_2, \cdots \) are independent and identically distributed with mean zero and variance 1. Then the random functions \( X_n \) defined by (1) satisfy

\[
(2) \quad X_n \overset{D}{\to} W^0.
\]

Here (2), and also similar relations later on, are interpreted in accordance with (4.5) and (4.7) of Billingsley’s book [2], depending on

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whether $W^0$ is construed as a random function or as a measure in the spirit of [2, p. 65]; the meaning is the same for the two interpretations. Since $h(x) = \sup_t |x(t)|$ with $x(t) = w(t) - tw(1)$ is a continuous function on $C$ in the sup-norm metric, (2) implies

$$\sup_t |X_n(t)| \overset{\mathcal{D}}{\to} \sup_t |W^0_t|,$$

an invariance principle, as statements like this are often called. Similarly,

$$\sup X_n(t) \overset{\mathcal{D}}{\to} \sup W^0_t, \quad \inf X_n(t) \overset{\mathcal{D}}{\to} \inf W^0_t.$$

For each $n$, let $\nu_n$ be a positive-integer-valued random variable defined on the same probability space as the $\xi_n$. Define $X_n$, a random element of $C$, as in (1), and $Y_n$, another random element of $C$, by

$$Y_n(t, \omega) = X_{\nu_n(t), \omega}(t, \omega).$$

**Theorem 1.** Suppose the random variables $\xi_1, \xi_2, \ldots$ are independent and identically distributed with mean zero and variance 1. If

$$\nu_n/n \overset{P}{\to} \nu,$$

where $\nu$ is a positive random variable, and

$$\frac{\xi_{\nu_n(t), \omega}(\omega)}{\nu_n(\omega)} \overset{P}{\to} 0, \quad \text{for every fixed } t,$$

then the random functions $Y_n$ defined by (3) satisfy

$$Y_n \overset{\mathcal{D}}{\to} W^0.$$

**Corollary 1.** Under the same assumptions as in Theorem 1 (6) implies

$$\sup_t |Y_n(t)| \overset{\mathcal{D}}{\to} \sup_t |W^0_t|,$$

$$\sup Y_n(t) \overset{\mathcal{D}}{\to} \sup W^0_t, \quad \inf Y_n(t) \overset{\mathcal{D}}{\to} \inf W^0_t.$$

**Remark 1.** Let $D$ be the space $D$ of Chapter 3 of P. Billingsley's book [2]. Define random elements $X_n^*, Y_n^*$ of $D$ by
(7) \[ X_n^*(t, \omega) = W_n(t, \omega) - tW_n(1, \omega), \]
(8) \[ Y_n^*(t, \omega) = X_n^*(\omega)(t, \omega) \]

with \( W_n(t, \omega) \) as in (1). Then Theorem B holds for \( X_n^* \) of (7) and, omitting condition (5), Theorem 1 holds for \( Y_n^* \) of (8). Also, in defining \( Y_n \) of (3) and \( Y_n^* \) of (8) it is not essential that the random variables \( \{\xi_n\} \) involved should be independent and identically distributed with unit variance. We have stated Theorem 1 here for random elements of \( C \) and for independent identically distributed random variables having unit variance only because it is, as will be shown later, directly applicable in this form to prove the random-sample-size Kolmogorov-Smirnov theorems. More general versions of Theorem 1 and detailed proofs of them will appear in [4]. We also note that for \( Y_n \) of (3) one postulates (5), for it is not true in general that \( \xi_{1:n}/n^{1/2} \rightarrow 0 \) and (4) imply (5).

For the proof of Theorem 1 we use Theorem B, Theorems 7.7, 8.1, 8.2 of P. Billingsley's book [2] and results of A. Rényi [7] and J. Mogyoródi [5]. First we show that for a single time point \( s \{X_n^*(s)\} \) is mixing with the normal distribution function \( \mathcal{N}(0, s(1-s)) \) in the sense of A. Rényi's definition of mixing sequences of events [7] and that it also satisfies the tightness condition of F. J. Anscombe [1]. Then, using Theorem B, Theorem 7.7 of [2] and Theorem 2 of [5], we show that the finite-dimensional distributions of \( Y_n \) of (3) converge to those of \( W^0 \). Next it is verified that the sequence \( \{Y_n\} \) is tight in the sense of Theorem 8.2 of [2] and then Theorem 1 follows from Theorem 8.1 of [2]. Details of this proof will appear in [4].

Let \( U_1, \ldots, U_n \) be independent random variables uniformly distributed on \([0, 1]\). The order statistics are defined as follows: \( Z_1 \) is the smallest, and so forth; \( U_{(n)} \) is the largest. Let

\[ F_n(t) = \frac{(\text{the number of the } U_i \leq t)}{n}, \quad t \in [0, 1]. \]

Define the Kolmogorov-Smirnov statistics

\[ D^+_n = n^{1/2} \sup_t (F_n(t) - t) = n^{1/2} \max_{k \leq n} (k/n - U_k^{(n)}), \]
\[ D^-_n = n^{1/2} \inf_t (F_n(t) - t) = n^{1/2} \min_{k \leq n} (k/n - U_k^{(n)}), \]
\[ D_n = n^{1/2} \sup_t \left| t - F_n(t) \right| = n^{1/2} \max_{k \leq n} \left| U_k^{(n)} - k/n \right|, \]

and the random-sample-size Kolmogorov-Smirnov statistics \( \Delta^+_n = D^+_n, \Delta^-_n = D^-_n, \Delta_n = D_n \).
Theorem 2. Under condition (4) of Theorem 1 we have

\[ \Delta_n^+ \Rightarrow \sup_t W_t^0, \quad \Delta_n^- \Rightarrow \inf_t W_t^0, \quad \Delta_n \Rightarrow \sup | W_t^0 |. \]

Proof of Theorem 2. Let \( S(n) = \xi_1 + \cdots + \xi_n, n = 1, 2, \cdots \) be the partial sum sequence of independent exponential random variables \( \{\xi_n\} \) with mean 1. L. Breiman [3, §13.6] obtained the following representation of \( D_n \)

\[ D_n = n^{1/2} \max_{k \leq n} \left| \frac{S(k)}{S(n + 1)} - \frac{k}{n} \right| \]

(9)

with analogous expressions for \( D_n^+ \) and \( D_n^- \). Here \( \Rightarrow \) means that the random variables in question have the same distribution. Put \( \xi_n = \xi_n - 1, S_k = S(k) - k \) and \( W_n(t, \omega) = S_{[nt]}(\omega)/n^{1/2} \). Then

\[ D_n = \sup_t | X_n^*(t, \omega) |, \quad \text{for } n \text{ large}, \]

(10)

\[ = \sup_t | X_n(t, \omega) |, \quad \text{for } n \text{ large}, \]

where \( X_n^* \) and \( X_n \) are respectively defined in terms of the above \( \xi_n \) and \( W_n \) via (7) and (1). Analogous asymptotic representations hold for \( D_n^+ \) and \( D_n^- \). The first asymptotic representation of (10) for \( D_n \) is true because \( E(\xi_n) = \sigma^2(\xi_n) = 1 \) and hence \( n/S(n + 1) \xrightarrow{b} \frac{\text{constant}}{1} \) and \( \xi_n + 1/n^{1/2} \xrightarrow{L} 0 \), while the second asymptotic representation of (10) is the consequence of \( \xi_{n+1}/n^{1/2} \xrightarrow{L} 0 \) uniformly in \( t \). The \( X_n \) of (10) satisfy the conditions of Theorem B and the usual Kolmogorov-Smirnov theorems follow. For \( \Delta_n \) we have (9) with \( n \) replaced by \( \nu_n \) on both sides. Now we show

\[ \Delta_n = \sup_t | Y_n^*(t, \omega) |, \quad \text{for } n \text{ large}, \]

(11)

\[ = \sup_t | Y_n(t, \omega) |, \quad \text{for } n \text{ large}, \]

where \( Y_n^* \) and \( Y_n \) are respectively defined in terms of the above \( \xi_n \) and \( W_n \) via (8) and (3); we also have the analogous asymptotic expressions for \( \Delta_n^+ \) and \( \Delta_n^- \). It is true in general that if \( \{Z_n\} \) is a sequence of random variables such that \( Z_n \xrightarrow{b} Z \) and \( \{\nu_n\} \) is a sequence of
positive-integer-valued random variables such that $\nu_n \xrightarrow{p} + \infty$, then $Z_{\nu_n} \xrightarrow{d} Z$. Now condition (4) of Theorem 1 implies $\nu_n \xrightarrow{p} + \infty$ and we have $n/S(n+1) \xrightarrow{a.s.} 1$. Consequently, $\nu_n/S(\nu_n+1) \xrightarrow{p} 1$. Using the fact that the $\zeta_n$ are exponential random variables with mean 1 and that $\nu_n \xrightarrow{p} + \infty$, it can be easily shown that $\zeta_{\nu_n+1}/\nu_n^{1/2}$ and $\xi_{[\nu_n]}^{(n)}/\nu_n^{1/2}$ both converge in probability to zero, the latter one uniformly in $t$. Hence both asymptotic representations of (11) are true. Also, given condition (4), the $Y_n$ of (11) satisfy the conditions of Theorem 1 and hence $Y_n \xrightarrow{d} Z$. The statements of Theorem 2 now follow from Corollary 1.

REMARK 2. Theorem 2 with $\nu = 1$ in (4) was proved by R. Pyke [6] in an interesting and different way, utilizing results about stochastic processes with two-dimensional parameter sets. We should also note that proving appropriate versions of Theorem 1, random-sample-size versions of the Kolmogorov-Smirnov theorems with weight functions like

$$f(t) = 1/t, \quad 1/(1 - t) \quad \text{and} \quad 1/[\mu(1 - t)]^{1/2},$$

which are important in applications, can also be proved in a similar way as well as two or more-sample random-sample-size versions. Statements and proofs for these results will also appear in [4].

REFERENCES


