1. **Introduction.** The problem of imbedding a closed differentiable manifold $M^n$ in a euclidean space can be weakened through the notion of (modulo 2) cobordism as follows. Is $M^n$ cobordant to a submanifold of $\mathbb{R}^{n+k}$? In this context we can prove an analogue, with improved dimensions, of H. Whitney's theorems [11], [12]. Let $\alpha(n)$ denote the number of ones in the binary expansion of $n$, and let $n > 1$.

**Theorem A.** Any $M^n$ is cobordant to a manifold $N^n$ that imbeds in $\mathbb{R}^{2n-\alpha(n)+1}$ and immerses in $\mathbb{R}^{2n-\alpha(n)}$.

For $n \neq 3$ this result is best possible as the examples below show. In some cases we can say more if certain Stiefel-Whitney numbers of $M^n$ are zero. Allow the empty set as a representative of the zero cobordism class. (Thus Theorem A holds for all $n$.)

**Theorem B.** (i) If $n$ is even ($n \neq 6$) and if $\bar{w}_{\alpha(n)} \cdot \bar{w}_{n-\alpha(n)}(M^n) = 0$ then $M^n$ is cobordant to a manifold $N^n$ that imbeds in $\mathbb{R}^{2n-\alpha(n)}$ and immerses in $\mathbb{R}^{2n-\alpha(n)-1}$.

(ii) If $n = 2^k$ or $2^k + 1$ and if $\bar{w}_i \cdot \bar{w}_{n-i}(M^n) = 0$ for $0 \leq i \leq n$ then $M^n$ is cobordant to a manifold $N^n$ that imbeds in $\mathbb{R}^{2n+1}$ and immerses in $\mathbb{R}^{2n+1-1}$.

Let $\mathcal{R}_*$ denote the modulo 2 cobordism ring, and let $MO(k)$ denote the Thom complex for $O(k)$. There are homomorphisms

$$\Phi(n, k): \pi_{n+k}(MO(k)) \to \mathcal{R}_n \quad \text{and} \quad \Psi(n, k, N): \pi_{n+k+N}(SN^kMO(k)) \to \mathcal{R}_n.$$

The image of $\Phi(n, k)$ is the set of cobordism classes that can be represented by submanifolds of $\mathbb{R}^{n+k}$ and hence coker $\Phi(n, k) = 0$ if $k > n - \alpha(n)$ by Theorem A. The image of $\Psi(n, k, N)(N \gg k)$ is the set of cobordism classes that can be represented by manifolds which immerge in $\mathbb{R}^{n+k}$ (see R. Wells [10]) and hence coker $\Psi(n, k, N) = 0$ if $k \geq n - \alpha(n)$, $N \gg k$.

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Real projective $n$-space $P^n$ ($n = 2^k + 1$, $k > 1$) is known not to imbed in $R^{2n-2}$ (see J. Levine [2]) but is cobordant to $S^n$ which does. Complex projective $n$-space $CP^n$ ($n = 2^k$, $k > 1$) does not immerse in $R^{4n-2}$ (see J. Levine [3]) but is cobordant to $P^n \times P^n$ which does. Hence in Theorem B it is sometimes necessary to have $M^n \not\cong N^n$. However we know of no manifold $M^n$ that does not imbed in $R^{2n-\alpha(n)+1}$ and immerse in $R^{2n-\alpha(n)}$.

2. Decomposables in $\mathcal{R}_*$. The main theorems are proved by imbedding and immersing manifolds constructed from real projective spaces until we have enough to form a basis of $\mathcal{R}_*$. We illustrate the method by outlining the proof of Theorem A.

**Proposition 2.1.** Suppose for each $n \geq 2^k - 1$ there is a manifold $V^n$ whose cobordism class $[V^n]$ is an indecomposable element of $\mathcal{R}_*$ and which imbeds in $R^{2n-\alpha(n)+1}$ and immerses in $R^{2n-\alpha(n)}$. Then Theorem A holds.

**Proof.** According to R. Thom [9] the cobordism classes $[V^n]$ generate the ring $\mathcal{R}_*$. Given a product $M^n = \prod V^j$ we can use the product immersion to immerse $M^n$ in $(\sum (2j - \alpha(j)))$-space. Because $\alpha(i+j) \leq \alpha(i) + \alpha(j)$ we have actually immersed $M^n$ in $(2n - \alpha(n))$-space or better. The product imbedding is not good enough, so to imbed $M^n$ in $(2n - \alpha(n)+1)$-space we use inductively the following well-known result. (For a three line proof see [7].)

**Lemma 2.2.** If $M^n$ imbeds in $R^s$, $N^n$ immerses in $R^t$, and $s+t > 2n$ (which is true if $m \geq n$) then $M^n \times N^n$ imbeds in $R^{s+t}$.

Any $M^n$ is cobordant to a disjoint union of products of the $V^j$ and we can imbed and immerse this disjoint union in the obvious way, thus proving Theorem A.

3. Construction of indecomposables. Let $n$ be even and let $n = r_1 + \cdots + r_k$ ($2 \leq r_1 < \cdots < r_k$) be the binary expansion of $n$ as a sum of distinct powers of 2. Thus $\alpha(n) = k$. Let $V^n = P^n$ if $k = 1$ and for $k > 1$ let $V^n$ be a submanifold of $K^{n+1} = P^{n+1} \times \prod_{i=1}^k P_{r_i}$ dual to $\alpha_1 + \cdots + \alpha_k \in H^1(K^{n+1}; \mathbb{Z})$ where $\alpha_i$ generates the modulo 2 cohomology ring of the $i$th factor.

**Proposition 3.1.** $[V^n]$ is an indecomposable element of $\mathcal{R}_*$ and $V^n$ satisfies the conditions of Proposition 2.1.

**Proof.** The first part follows from a computation of the total Stiefel-Whitney class $w(V^n)$ and from standard arguments using elementary symmetric functions (see R. E. Stong [8, p. 79]). The second
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part is based on an immersion of $P^n (n = 2^r + 1)$ in $R^{2n-8}$ due to B. J. Sanderson [6]. Whitney’s results ($M^n$ imbeds in $R^{2n}$ and immerses in $R^{2n-1}$) and the product immersion or inductive use of Lemma 2.2 finish the proof.

REMARK 3.2. $M^n = \prod_{i=1}^{2^r} P_{2^i}$ has $\bar{w}_k \cdot \bar{w}_{n-k} \neq 0$ and hence furnishes a counterexample to improving Theorem A when $n$ is even.

The above construction of even dimensional generators was inspired by the work of J. Milnor [5] and the following is a modification of A. Dold’s construction of odd dimensional generators of $\mathfrak{N}_{\mathfrak{K}}$ [1]. Given a positive integer $m$ and a topological space $X$ form $P(m, X)$ from $S^m \times X \times X$ by identifying $(u, x, y)$ with $(-u, y, x)$.

**PROPOSITION 3.3.** $P(m, M^n)$ is an $(m+2n)$-manifold and represents an indecomposable element of $\mathfrak{N}_{\mathfrak{K}}$ if and only if $[M^n]$ is indecomposable and the binomial coefficient $\binom{m+n}{n} \equiv 1 \pmod{2}$.

A map $X \to Y$ induces a map $P(m, X) \to P(m, Y)$ and differentiable imbeddings and immersions are preserved by this functor. Also $P(m, R^n)$ is the total space $E(s\gamma_m \oplus s\epsilon)$ where $\gamma_m$, $\epsilon$ are respectively the canonical line bundle and the trivial line bundle over $P^m$. Thus we have proved

**PROPOSITION 3.4.** If $M^n$ imbeds (immerses) in $R^t$ and $E(s\gamma_m \oplus s\epsilon)$ imbeds (immerses) in $R^t$ then $P(m, M^n)$ imbeds (immerses) in $R^t$.

Now let $n$ be odd, $n \neq 2^k - 1$. We can write uniquely $n = 2^r (2^s + 1) - 1 = 2r - 1 + 2^{r+1} s \ (r > 0, s > 0)$. Let $a = 2^r - 1$, $b = 2^s r$ and $V^n = P(a, V^t)$.

**PROPOSITION 3.5.** $V^n$ satisfies Proposition 2.1.

**PROOF.** By Propositions 3.1 and 3.3, $[V^n]$ is indecomposable. Using the imbedding and immersing part of Proposition 3.1 we can apply Proposition 3.4 to reduce the proof to imbedding and immersing certain sums of line bundles over $P^n$. Now the work of M. Mahowald and R. Milgram [4, Lemma 1.5] gives the required result.

**REMARK 3.6.** Using the notation of the beginning of this section let $M^{n+1} = P(1, \frac{1}{2} r_k) \times \prod_{i=1}^{2^r} P_{2^i}$. If $n > 2$ then $w_{k+1} \cdot \bar{w}_{n-k-1} (M^n) \neq 0$ so $M^{n+1}$ serves as a counterexample to improving Theorem A.

**REFERENCES**


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