ON THE SEMISIMPLICITY OF INTEGRAL REPRESENTATION RINGS

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For a finite group $G$ and a ring $R$, define the integral representation ring $\mathcal{A}(RG)$ as the abelian group generated by the isomorphism classes of $RG$-lattices, with

$$[M] + [M'] = [M \oplus M'],$$

and

$$[M][M'] = [M \otimes_R M'].$$

The integral representation algebra $\mathcal{A}(RG)$ is $C \otimes_R \mathcal{A}(RG)$. When does $\mathcal{A}(RG)$ contain nontrivial nilpotent elements?

Let $|G| = p^a n$, where $p \nmid n$, $p$ prime. Denote by $Z_p$ the $p$-adic valuation ring in $\mathbb{Q}$, and by $Z_p^*$ its completion. Reiner has shown

(i) If $\alpha = 1$, then $\mathcal{A}(Z_pG)$ and $\mathcal{A}(Z_p^*G)$ have no nonzero nilpotent elements (see [1]).

(ii) If $\alpha \geq 2$, and $G$ has an element of order $p^2$, then both $\mathcal{A}(Z_pG)$ and $\mathcal{A}(Z_p^*G)$ contain nonzero nilpotent elements (see [2]).

We have been able to settle the open case as to what happens when $G$ has a $(p, \ell)$-subgroup. Our main result is

**Theorem 1.** Whenever $\alpha > 1$, both $\mathcal{A}(Z_pG)$ and $\mathcal{A}(Z_p^*G)$ contain nonzero nilpotent elements.

As a matter of fact, the construction used shows

**Theorem 2.** If $|G|$ is not squarefree, then $\mathcal{A}(ZG)$ and $\mathcal{A}(Z'^G)$ contain nonzero nilpotent elements, where

$$Z' = \{ a/b : a, b \in Z, b \text{ coprime to } |G| \}.$$

In the other direction, Reiner proved

(iii) If $|G|$ is squarefree, then $\mathcal{A}(Z'G)$ has no nonzero nilpotent elements (see [1]).

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All of our results generalize to the case where $\mathbb{Z}$ is replaced by the ring of algebraic integers in an algebraic number field.

As a consequence of Theorem 1, we have Theorem 3. Let $k$ be a field of characteristic $p$, $p$ an odd prime. If $G$ has a noncyclic $p$-Sylow subgroup, $a(kG)$ contains nonzero nilpotent elements.

References


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