SMOOTH MAPS TRANSVERSE TO A FOLIATION

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1. Introduction. This article presents a Smale-Hirsch-type classification theorem for smooth maps transverse to a foliation. Let $M, W$ be smooth manifolds, with tangent bundles $TM, TW$, and let $\text{Hom}(M, W), \text{Hom}(TM, TW)$ represent the spaces of smooth maps $M \to W$ and of fibrewise linear maps $TM \to TW$, where we give to $\text{Hom}(TM, TW)$ the compact-open topology, and to $\text{Hom}(M, W)$ the $C^1$-compact-open topology; thus the map $d: \text{Hom}(M, W) \to \text{Hom}(TM, TW)$, which associates to each smooth map its differential, is continuous.

Suppose $W$ carries a foliation $\mathcal{F}$, and let $T\mathcal{F}$ denote the subbundle of $TW$ tangent to $\mathcal{F}$ (i.e. the embedding $T\mathcal{F} \to TW$ is an integrable distribution). Let $\text{Trans}(TM, T\mathcal{F})$ be the subspace of $\text{Hom}(TM, TW)$ consisting of those maps fibrewise transverse to $T\mathcal{F}$, and let

$$\text{Trans}(M, \mathcal{F}) = d^{-1} \text{Trans}(TM, T\mathcal{F}) \subset \text{Hom}(M, W).$$

**Theorem 1.** If $M$ is open, then the differential map $d: \text{Trans}(M, \mathcal{F}) \to \text{Trans}(TM, T\mathcal{F})$ is a weak homotopy equivalence.

Suppose now $W$ has a Riemannian metric, so we can define $N\mathcal{F}$, the normal bundle to $\mathcal{F}$, to be the bundle whose fibre at $x \in W$ is the orthogonal complement to $T\mathcal{F}_x$. Then the space $\text{Epi}(TM, N\mathcal{F})$ of fibrewise linear and surjective maps $TM \to N\mathcal{F}$ is a subspace and, in fact, a deformation retract, of $\text{Trans}(TM, T\mathcal{F})$. If we let $p: \text{Hom}(TM, TW) \to \text{Hom}(TM, TW)$ be composition with fibrewise orthogonal projection of $TW$ onto the sub-bundle $N\mathcal{F}$ then Theorem 1 has the immediate corollary:

**Theorem 2.** If $M$ is open, then the map $p \circ d: \text{Trans}(M, \mathcal{F}) \to \text{Epi}(TM, N\mathcal{F})$ is a weak homotopy equivalence.

**Remarks.** Theorem 1, which was proposed to the author by J. W. Milnor, has a special case (where $\mathcal{F}$ = the foliation by points) the

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author's classification of submersions [6, Theorem A]. This theorem, as well as the rest of the Smale-Hirsch-type theorems for open manifolds, also follows from a general theorem proved by M. L. Gromov in his dissertation [2].

As an application, let us give a short proof of the following result from [7].

**Theorem 3.** Let \( \sigma \) be a \( q \)-plane field on an open manifold \( M \). If the structural group of \( \sigma \) considered as a \( q \)-plane bundle can be reduced to a discrete group, then \( \sigma \) is homotopic to the \( q \)-plane field normal to a foliation.

**Proof** (see also [2]). Let \( S \) be the total space of the bundle \( \sigma \); by a theorem of Ehresmann [1], [3], \( S \) has a foliation \( \mathcal{F} \) of codimension \( q \) of which the zero cross-section is a leaf. Orthogonal projection: \( TM \rightarrow \sigma \) can be interpreted as an element \( H_0 \) of \( \text{Epi}(TM, N\mathcal{F}) \) via the usual identification of \( \sigma \) with the tangent space to the fibres of \( S \) along the zero cross-section. Theorem 2 implies that \( H_0 \) is homotopic in \( \text{Epi}(TM, N\mathcal{F}) \) to \( H_1 = p \circ df \), where \( f \in \text{Trans}(M, \mathcal{F}) \). If \( t \rightarrow H_t \) is the homotopy then \( t \rightarrow (\text{orthogonal complement of ker } H_t) \) gives a homotopy between \( \sigma \) and the \( q \)-plane field normal to the pulled-back foliation \( f^* \mathcal{F} \).

Theorem 1 has also been applied to the study of classifying spaces for foliations [4], [5].

2. **Outline of proof of Theorem 1.** The proof follows the lines of the proof of the submersion theorem of [6]. This method of proof can be summarized as follows. In order to compare the spaces \( \text{Trans}(M, \mathcal{F}) \) and \( \text{Trans}(TM, T\mathcal{F}) \), we consider the pair of spaces \( \text{Trans}(U, \mathcal{F}) \) and \( \text{Trans}(TU, T\mathcal{F}) \) for each closed submanifold-with-boundary \( U \subset M \), with \( \dim U = n = \dim M \). The assignments \( U \rightarrow \text{Trans}(U, \mathcal{F}), \quad U \rightarrow \text{Trans}(TU, T\mathcal{F}) \) can be thought of, following Gromov, as contravariant functors from the category \( C_M \) of closed \( n \)-dimensional submanifolds-with-boundary of \( M \) and inclusion maps, to the category \( \mathcal{F} \) of topological spaces.

**Definition.** A functor \( A : C_M \rightarrow \mathcal{F} \) will be called admissible if it has the following properties.

(a) \( A \) is locally defined, in that if \( U_1 \cap U_2 \subset C_M \), and if \( X \subset A(U_1) \times A(U_2) \) is defined by \( X = \{(f_1, f_2), f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2} \} \), then the natural map \( A(U_1 \cup U_2) \rightarrow X \) is a homeomorphism.

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1 If \( U \subset M \) is a manifold with boundary, we define \( f \in \text{Trans}(U, W) \) to mean that \( f \) extends to a transverse map of an open neighborhood of \( U \) in \( M \), and \( TU \) to be \( TM|_U \). These manifolds with boundary may have corners, as described in [6, §0].
(b) If $V \supset U$ is a collarlike neighborhood of $U$ (see [6, p. 176] for a precise definition) then the restriction map $A(V) \to A(U)$ is a weak homotopy equivalence, and has the covering homotopy property.

(c) If $V = U$ with a handle of index $\lambda$ attached, then the restriction map $A(V) \to A(U)$ has the covering homotopy property. If $M$ is open and $n$-dimensional then this property need only be satisfied for handles of index $\lambda \leq n - 1$.

**Proposition (Smale-Thom-Hirsch-Palais-Haefliger-Poenaru Theorem Proving Machine).** Let $A, B : \mathcal{C}_M \to \mathcal{C}$ be admissible functors, and let $\Phi : A \to B$ be a natural transformation. If $\phi : A(D^n) \to B(D^n)$ is a weak homotopy equivalence for each embedded $n$-disc $D^n \subset M$, then so is $\Phi : A(M) \to B(M)$.

**Proof.** See [6, §6].

Theorem 1 will follow from this Proposition once it is shown that, on an open manifold, $\text{Trans}(\_ , \mathfrak{F})$ and $\text{Trans}(T , T\mathfrak{F})$ are admissible functors, and that $d : \text{Trans}(D^n , \mathfrak{F}) \to \text{Trans}(TD^n , T\mathfrak{F})$ is a homotopy equivalence. Most of this is a straightforward generalization of the corresponding lemmas for submersions. The only point that seems to require new analysis is showing that $\text{Trans}(\_ , \mathfrak{F})$ has property (c). This is treated in the next two sections.

3. The covering homotopy property. The proof of this property in the submersion case involves a long, geometric argument [6, §4]; examination of this argument shows that it uses only the following facts about submersions:

(a) submersions are stable in the sense of [6, Lemma 3.1];
(b) submersions form an open and locally defined subspace of $\text{Hom}(M, W)$;
(c) if $f : M \to W$ is a submersion and $h$ is a diffeomorphism of $M$, then $f \circ h$ is a submersion.

Facts (b) and (c) are clearly also true of maps transverse to a foliation. (It turns out that facts (b) and (c) alone are sufficient, and that if $M$ is open the appropriate Smale-Hirsch type theorem holds for any subspace of $\text{Hom}(M, W)$ satisfying these two conditions. This observation is due to Gromov [2].) In order to use the "good position" method of proof, it remains to establish an analogue to the stability lemma; the statement is below.

Let $U \subset M$ be a compact manifold-with-boundary, and suppose

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* If $M$ is not compact, let $A(M)$ be the inverse limit of $A(U_i)$ where $U_i \in \mathcal{C}_M$ $U_i \subset U_{i+1}$ and $\bigcup U_i = M$. 
given $f \in \text{Trans}(U, \mathcal{F})$. By definition, $f$ extends to a transversal map of an open neighborhood of $U$. In particular, we may suppose that $f = \tilde{f} | U$, where $\tilde{f} \in \text{Trans}(L, \mathcal{F})$ and $L \subset M$ is a compact manifold-with-boundary, $U \subset \text{Int } L$. Let $E$ be the total space of $\tilde{f}^* T\mathcal{F}$, let $\beta : E \to T\mathcal{F}$, $\pi : E \to L$ be the canonical maps, and let $\mathcal{G}$ be the foliation of $E$ by fibres.

**Local factoring lemma.** With data as above, there exist

1. an open tubular neighborhood $N$ of the zero cross-section in $E$ (we will consider $N$ as an open manifold with boundary $\partial N = \pi^{-1} \partial L$);
2. a submersion $\phi : N \to W$ with $d\phi(T\mathcal{G}) \subset T\mathcal{F}$;
3. a neighborhood $\eta$ of $f$ in $\text{Hom}(U, W)$;
4. a continuous map $\nu : \eta \to \text{Aut}(N, \mathcal{G})$ (the space of foliation-preserving diffeomorphisms of $N$ which are the identity near $\partial N$) such that $\nu\tau = \text{id}$ and such that $g = \phi \circ \nu|U$ for $g \in \eta$.

**Remarks.** Roughly speaking, this lemma means that $f$ can be extended to a submersion $\phi$ of a larger manifold $N$ in such a way that maps nearby to $f$ can be obtained by composing $\phi$ with leaf-preserving diffeomorphisms of $N$ nearby to the identity, and which are equal to the identity near $\partial N$. Condition (2) implies that if a map $h$ is transversal to $\mathcal{G}$, then $\phi \circ h$ will be transversal to $\mathcal{G}$.

This lemma is proved in the next section. Let us now see how it is used to lift an arc of maps from $\text{Trans}(U, \mathcal{F})$ to $\text{Trans}(V, \mathcal{F})$. The technical details involved in lifting a homotopy of a cube of dimension $>0$ are completely analogous to those for the submersion case. The pictures in [6, §4], which illustrate the special case of this argument for $\mathcal{F}$ the foliation by points, should be consulted.

**Lifting an arc.** Suppose $F_0 \in \text{Trans}(V, \mathcal{F})$ and that $f_t$, $0 \leq t \leq 1$, is a homotopy of $f_0 = F_0 | U$. Each $f_t$ has a neighborhood $\eta_t$ as described above; clearly, we may suppose that $f([0, 1]) \subset \eta_0$. Let $\phi : N \to W$ be the submersion corresponding to $\eta_0$, and let $\nu : \eta_0 \to \text{Aut}(N\mathcal{G})$ be as in the local factoring lemma.

We define a collar neighborhood $C$ of $U$ in $V$ to be a neighborhood diffeomorphic to $U \cup \hat{U} \times [0, 1]$, where $\hat{U} \cong S^{n-1} \times D^{n-1}$ is the attaching surface of the handle. Let $\hat{C}$ be the boundary of $C$ in $V$ (see [6, Figure 4.5].)

We say that $F_0$ is in good position with respect to $\phi$ if we can find a collar neighborhood $C$ of $U$ in $V$ and an embedding $\beta : C \to N$ such that

1. $\beta | U$ is the zero cross-section;
2. $\phi \circ \beta = F_0 | C$;
3. $\beta(\hat{C}) \subset \partial N$;
4. $\beta$ is transverse to $\mathcal{G}$. 

If $F_0$ is in good position with respect to $\phi$, then the arc $f_i$ can be lifted to $\text{Trans}(V, \mathcal{F})$ by defining

$$F_t(x) = F_0(x), \quad x \in V - C,$$

$$= \phi \circ \nu_t \circ \beta(x), \quad x \in C.$$

Otherwise we remark that $F_0$ is in good position with respect to $\phi|S$, where $S \subset N$ is some smaller tubular neighborhood. This follows from comparing the maps $F_0|L \cap V$ and $\phi|L \cap V$ (where $L \subset N$ as the zero cross-section). These maps agree on $U$, so they are close near $U$, so since $\phi$ is a submersion there exists, by [6, Lemma 3.1], an embedding $\kappa: \bar{U} \to N$, where $\bar{U} \subset L \cap V$, $\bar{U} \cong U \cup \bar{U} \times [0, 1]$ is a collar neighborhood of $U$ in $V$, such that $\phi \circ \kappa = F_0|\bar{U}$, and $\kappa|U$ is the zero cross-section. If $\bar{U}$ is chosen small enough $\kappa$ will be transverse to $\mathcal{G}$. Then pick $L'$ such that $U \subset L'$ and $\pi \circ \kappa(\bar{U}) \cap \partial L' = \pi \circ \kappa(\bar{U} \times \{1\})$. Let $S = \pi^{-1}L' \cap N$. Then taking $C = U \cup \bar{U} \times [0, \frac{1}{2}]$ and $\kappa|C:C \to S$ shows that $F_0$ is in good position with respect to $\phi|S$.

Now pick an $\epsilon > 0$ such that $\nu_t(U) \subset S$ for $t \leq \epsilon$ and such that the arc of embeddings $\nu_t: U: U \to S$ can be realized by composing the zero cross-section with an arc $\sigma$, in $\text{Aut}(S, \mathcal{G})$. Then the argument above shows how to lift $f([0, \epsilon])$ to an arc $\bar{F}: [0, \epsilon] \to \text{Trans}(V, \mathcal{F})$ starting at $F_0$.

In order to continue past $\epsilon$ we change $\bar{F}$ to a new lifting $\tilde{F}$ such that $\tilde{F}_t$ is in good position with respect to $\phi \circ \nu$. Briefly, this is done by a $\mathcal{G}$-transverse isotopy of $\kappa|U \times \{\frac{1}{2}\}$, keeping ends fixed, in a larger tubular neighborhood $\bar{N}$. The isotopy described in [6, Sublemma 4.6] may be performed on $\pi \circ \kappa$ and lifted under $\pi$, starting at $\kappa$, to give the desired arc of maps.

4. Proof of local factoring lemma.

Proof. (With notation from §3). Pick a connection in $TW$ such that if $v \in T\mathcal{F}$, the arc $t \to \exp(tv)$ lies in a leaf, and define $\phi:E \to W$ by

$$\phi(v) = \exp_{\tilde{f}((\pi \circ \sigma))}(v).$$

It follows from transversality of $\tilde{f}$ that this $\phi$ is a submersion along $L$ (which we identify with the zero cross-section in $E$) and therefore on some tubular neighborhood $N$ of $L$ in $E$, e.g. $\{|v| < \epsilon\}$ for some $\epsilon > 0$. This is essentially a "foliated tubular neighborhood," as described in [8, Proposition 3.1]. Observe that by the choice of connection, $d\phi(T\mathcal{G}) \subset T\mathcal{F}$, as required.

Suppose $g \in \text{Hom}(U, W)$ is $C^1$-close to $f$. Then since $\phi$ is a submersion there is an embedding $\mu_\phi: U \to N$ such that $\phi \circ \mu_\phi = g$; in fact the argument of [6, Lemma 3.1] gives a continuous map $\mu: \eta$
Emb(C\gamma, N) (where \eta is a neighborhood of f in Hom(U, W) and Emb(U, N) is the space of embeddings) such that \mu_f = the zero cross-section and \phi \circ \mu_g = g for g \in \eta. If g is close enough to f, \mu_g will also be transverse to the fibres of g, and will extend to an embedding: L \rightarrow N transverse to the fibres and equal to the zero cross-section near \partial L. This embedding in turn will extend to a fibre-preserving diffeomorphism \nu_g of N, which leaves a neighborhood of \partial N = \pi^{-1}\partial L fixed; it is easy to check that these extensions can be defined so as to depend continuously on g.

**References**


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