ON THE COHOMOLOGY OF STABLE TWO
STAGE POSTNIKOV SYSTEMS

BY JOHN R. HARPER

Communicated by Norman Steenrod, February 2, 1970

Introduction. Let $\xi = (E, p, B, F)$ denote a two stage Postnikov system with stable $k$-invariant. We announce results about $H^*(O\xi)$ as a Hopf algebra over the Steenrod algebra. Mod 2 cohomology is used exclusively. Unexplained notation is from [4] and [5]. I am grateful to D. Anderson, W. Massey, F. Peterson and H. Salomonsen for many useful remarks.

We make the following assumptions on $\xi$, in addition to those of [5, p. 38]. $F$ and $B$ are simply connected products of finitely many Eilenberg-MacLane spaces. The nonzero homotopy groups of the factors of $B$ are infinite cyclic or cyclic of order $2^k$, $k = 1, 2, \cdots$. All factors of $F$ have $Z_2$ (cyclic group of order 2) as their only nonzero homotopy group.

Results of [3], [4], [5] and [8] give $H^*(O\xi) \cong R \otimes U(X')$. The isomorphism is as algebras over $Z_2$ and $\otimes$ is over $Z_2$. $R = R(O\xi)$ is considered as known, [5, p. 54]. In general $H^*(O\xi)$ does not split this way as a Hopf algebra over $Z_2$. The new result is Theorem A. It gives $H^*(O\xi)$ as a coalgebra over $R$. It also gives information on the extension problem represented by the fundamental sequence of $O\xi$ [5, p. 54]. This use of the Hopf algebra structure is well known, [1], [5] and [6].

1. The main theorem. Consider the following diagram of unstable $A$-modules and $A$-maps. The squares are commutative.

\[\begin{array}{ccc}
X'(O\xi) & \leftarrow & Y''/\lambda Y'' \\
\alpha & \pi & c
\end{array}\]

\[\begin{array}{ccc}
Y & \overset{f^*}{\longrightarrow} & Z \\
\rho
\end{array}\]

\[\begin{array}{ccc}
\sigma_{B_6} & \downarrow & \sigma_B & \downarrow & \sigma' & \downarrow \\
\Omega Y & \overset{\Omega f^*}{\longrightarrow} & \Omega Z & \longrightarrow & \Omega Z'
\end{array}\]

Here $\alpha$ is an $A$-isomorphism of degree $-1$; $\pi$, $\rho$ and $\rho'$ are natural projections; $c$ is inclusion, and $\sigma'$ is the obvious map. The remaining

AMS Subject Classifications. Primary 5550; Secondary 5534.

Key Words and Phrases. Postnikov system, stable $k$-invariant, Steenrod algebra, Hopf algebra, Massey-Peterson fundamental sequence.

807
maps and modules are as in [5, p. 63]. In particular \( Y'' = \ker \Omega f^*, Z' \) and \( \Omega Z' \) are coker \( f^* \) and coker \( \Phi f^* \) respectively.

Using (1) and (2) we associate with each homogeneous element \( x \in X'(\Omega \xi) \) an element \( w \in \Omega Z' \) as follows. Let \( y \in \Omega Y \) such that \( \alpha^*(y) = x \). Let \( t \in Y \) such that \( \sigma_{B^1}(t) = y \). Since \( \sigma_{B^1} f^*(t) = \lambda z \) for some \( z \in Z \). Let \( w = \sigma' \rho (z) \). Note that the calculation of \( w \) just involves the Adem relations.

**Proposition 1.** \( w \) is a unique element of \( \Omega Z' \).

**Proof.** \( \sigma_{B^1} \) is a map of degree \(-1\) and \( \lambda \) doubles degrees. Hence \( \rho f^* \sigma_{B^1}(\lambda Y'') = 0 \). This and looking at the choices involved in the definition of \( w \) give the result.

**Theorem A.** There exists an element \( e \in P(\Omega \xi) \) such that \( \Omega i^*(e) = x \) and \( \bar{\mu}_2(e) = \rho(w \otimes w) \). Here \( \rho \) is the map

\[
\Omega f^* \otimes \Omega f^*: R(\Omega \xi \times \Omega \xi) \to P(\Omega \xi \times \Omega \xi).
\]

The notation is [5, p. 63]. The proof uses the Serre spectral sequence in a manner similar to but more involved than arguments of [2] and [7].

**Remarks.**

1. Theorem A amounts to calculating the homomorphism

\[
X'(\Omega \xi)/\text{im } \sigma_i \to R(\Omega \xi \times \Omega \xi)/\text{im } \bar{\mu}_1
\]

in the exact sequence at the bottom of p. 63 [5]. (\( \mu_i \) should be replaced by \( \bar{\mu}_i \) if \( i = 1, 2 \).)

2. If degree \( x \) is odd, then \( w = 0 \). If degree \( x \) is even, it is quite possible for \( w = 0 \) and not have \( x \in \text{im } \sigma_1 \). An example is given by \( B = K(Z_2, 2) \), \( F = K(Z_2, 7) \) and \( k \)-invariant \( Sq^4 Sq^2 \). This example was also discovered by Massey.

3. Let \( \{x_i\} \) be a homogeneous \( Z_2 \)-basis for \( X'(\Omega \xi) \). Let \( \{e_i\} \subset P(\Omega \xi) \) satisfy Theorem A with \( \Omega i^*(e_i) = x_i \). Then, by results of [4] and [5], \( \{1\} \cup \{e_i\} \) form a simple system of generators for \( H^*(\Omega E) \) as an algebra over \( R \). Thus Theorem A calculates the coproduct of \( H^*(\Omega E) \) considered as coalgebra over \( R \). (\( R \) acts on \( H^*(\Omega E) \otimes H^*(\Omega E) \) via \( \rho \).

4. Let \( \{x_i\} \) and \( \{e_i\} \) be as in Remark 3. Let \( \theta \in A \) and consider \( \sum x_j = \Omega i^*(\theta e_i) \). Then \( \langle \theta e_i + \sum e_i \rangle = \Omega \rho^*(r) \) for a unique \( r \in R \). The naturality of fundamental sequences with respect to loop multiplication and suspension gives much information about \( r \). For example a unique element \( [r] \in R/S \) is determined by the formula

\[
q \bar{\mu}_1([r]) = \mu_3(\theta e_i + \sum e_i).
\]
Here $S \subset R$ is the $A$-submodule of primitives and $\mu_1 : R/S \to R \otimes R$ is considered as an $A$-map. It is well known to be a monomorphism. A similar formula can be obtained using suspension. We remark that if $F$ and $B$ are 2-connected and $R$ is an exterior algebra over $\mathbb{Z}$, then such formulae and the knowledge of $P(\Omega^2 \xi)$ as an $A$-module permit a complete calculation of $P(\Omega \xi)$ as an $A$-module. We defer details to a longer paper.

**REFERENCES**


THE UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627