GENERALISED NUCLEAR MAPS IN NORMED LINEAR SPACES

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1. Preliminary definitions and notations. Grothendieck [3] and Pietsch [6] present an exhaustive study of nuclear operators and nuclear maps. The notion of a nuclear operator was extended by Persson and Pietsch in a recent paper [5] and they study in detail the \( p \)-nuclear and quasi-\( p \)-nuclear maps. In this paper we define and study certain linear maps called \( \lambda \)-nuclear and quasi-\( \lambda \)-nuclear maps. Our definition and generalisation here are motivated by the Köthe sequence spaces and their duality theory. For the special case \( \lambda = l^1 \) we obtain the nuclear operators and for \( \lambda = l^p \) we obtain the \( p \)-nuclear maps; also, the special case \( \lambda = c_0 \) yields the \( \infty \)-nuclear operators of Persson and Pietsch. Most of the results in this work are motivated by the work of Persson and Pietsch [5] and Köthe sequence spaces.

We shall briefly outline our assumptions. For definitions not stated here see Garling [1], Köthe [4], Ruckle [7], Sargent [9] and Zeller [10]. Let \( \lambda \) be a symmetric sequence space of scalars and \( \lambda^* \) be its Köthe dual. We shall assume that \( \lambda \) is provided with the Mackey topology of the duality \( (\lambda, \lambda^*) \) and that this topology is provided by a norm \( \rho \), \( \rho \) itself being an extended seminorm on \( \omega \). We assume now that \( \lambda \) is solid and that it is \( K \)-symmetric, i.e., for each \( x \in \lambda \) and for each permutation \( \pi \) of \( I^+ \) we have \( x_\pi \in \lambda \) and \( \rho(x) = \rho(x_\pi) \). \( \lambda \) is also assumed to be a BK space with AK. We remark that our assumptions imply that \( \lambda = \omega \) or \( \lambda = l^\infty \) or \( \lambda \subseteq c_0 \). The space \( \lambda^* \) is now considered as the topological dual of \( \lambda \) and equipped with its natural norm topology.

We pause now to point out that in addition to the spaces \( l^p \), \( 1 \leq p < \infty \), the sequence spaces \( n(\phi) \) of Sargent [8] and the sequence spaces \( \mu_{a,p} \) and \( \nu_{a,p} \) of Garling [2] serve as examples of the type of sequence spaces \( \lambda \) we consider. Garling shows also that his spaces \( \mu_{a,p} \) are in general not linearly homeomorphic to \( l^p \).

Next let \( E \) and \( F \) be normed linear spaces. Then \( \lambda(E) \) is the (vector sequence) space of all vectors \( x = (x_n) \), \( x_n \in E \) for each \( n \) and such that the sequence \( (\langle x_n, a \rangle) \in \lambda \) for each \( a \in E^* \). Formally define

\[
\varepsilon_{\lambda}(x) = \sup_{\|a\| \leq 1} \rho(\| \langle x_n, a \rangle \|),
\]

where \( \rho \) is the norm on \( \lambda \).

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\( \lambda[E] \) is the space of sequences \( x = (x_n), x_n \in E \) for each \( n \) and such that \( \langle |x_n| \rangle \in \lambda \); the space \( \lambda[E] \) is equipped with a natural norm topology given by \( \|x\|_\lambda = \rho(\langle |x_n| \rangle) \).

2. \( \lambda \)-nuclear maps. Let \( T \) be a linear map on the normed space \( E \) into another, \( F \). We define \( T \) to be a \( \lambda \)-nuclear map if \( T \) admits the representation

\[
Tx = \sum_{n=1}^{\infty} \langle x, a_n \rangle y_n, \quad x \in E,
\]

where \( a = (a_n) \in \lambda[E^\prime] \) and \( y = (y_n) \in \lambda^*(F) \) with \( \varepsilon_\lambda(y) < \infty \). There may be other representations of \( T \) in the above form. Keeping this in mind, we define

\[
N_\lambda(T) = \inf \{ \|a\|_{\lambda^*} \cdot \varepsilon_\lambda(y) \}
\]

where the infimum is taken over all possible representations of \( T \) in the above form.

We observe that \( \lambda \)-nuclear maps can be defined in the following equivalent way: say \( T \) is \( \lambda \)-nuclear if \( T \) has the representation

\[
Tx = \sum_{n=1}^{\infty} \alpha_n \langle x, u_n \rangle y_n,
\]

where \( \|u_n\| \leq 1 \) for each \( n \), \( \alpha = (\alpha_n) \in \lambda \) and \( y = (y_n) \in \lambda^*(F) \) with \( \varepsilon_\lambda(y) \leq 1 \). In this case

\[
N_\lambda(T) = \inf \rho(\alpha).
\]

Let \( N_\lambda(E, F) \) denote the set of all \( \lambda \)-nuclear maps on \( E \) into \( F \).

**Theorem 1.** Each \( \lambda \)-nuclear map \( T \) is continuous and \( \|T\| \leq N_\lambda(T) \).

**Theorem 2.** \( N_\lambda(E, F) \) is a quasi-normed linear space under the norm \( N_\lambda \); also if \( F \) is a Banach space \( N_\lambda(E, F) \) is complete if \( \lambda \) is made of all sequences \( u \in \omega \) for which \( \rho(u) < \infty \).

**Theorem 3.** If \( A(E, F) \) denotes the space of all operators \( T \) on \( E \) which have finite dimensional ranges in \( F \), then \( A(E, F) \) is a dense subspace of \( N_\lambda(E, F) \).

**Corollary.** If \( F \) is a Banach space then each \( T \in N_\lambda(E, F) \) is a compact linear map and each such \( T \) has a separable range space.

The next two theorems play an important role in the factorization theorem (Theorem 6) characterizing \( \lambda \)-nuclear maps.
Theorem 4. Let $E$, $F$ and $G$ be normed linear spaces. Then we have the following:
(a) If $T \in N_\lambda(E, F)$ and $S \in L(F, G)$ then $S \circ T \in N_\lambda(E, G)$ and $N_\lambda(S \circ T) \leq \|S\| \cdot N_\lambda(T)$.
(b) If $T \in L(E, F)$ and $S \in N_\lambda(F, G)$ then $S \circ T \in N_\lambda(E, G)$ and $N_\lambda(S \circ T) \leq N_\lambda(S) \cdot \|T\|$. 

Theorem 5. Let $\delta = (\delta_n)$ be a fixed member of $\lambda$. Then the map $D: l^\infty \to \lambda$ defined by $D(u) = (u, \delta)$ is a $\lambda$-nuclear map and $N_\lambda(D) = \rho(\delta)$.

Theorem 6. Suppose $F$ is a Banach space. Then the map $T \in L(E, F)$ is $\lambda$-nuclear if and only if it can be factorized as follows:

$$T = Q \circ D \circ P, \quad E \to l^\infty \to \lambda \to F$$

where $P$ and $Q$ are continuous linear maps with $\|P\| \leq 1$ and $\|Q\| \leq 1$ and $D$ is as defined in Theorem 5.

3. Quasi-$\lambda$-nuclear maps. A linear map $T$ on $E$ into $F$ is defined to be quasi-$\lambda$-nuclear if there exists a sequence $a = (a_n)$ of elements of $E'$ such that $a \in \lambda[E']$ and $\|Tx\| \leq \rho(\langle a, x \rangle)$ for each $x \in E$. Set $Q_\lambda(T) = \inf \|a\|_\sigma$, where the infimum is taken over all admissible $a$. Then one can prove that $Q_\lambda(E, F) \subseteq L(E, F)$ with $\|T\| \leq Q_\lambda(T)$. Also $N_\lambda(E, F) \subseteq Q_\lambda(E, F)$ with $Q_\lambda(T) \leq N_\lambda(T)$ for $T \in N_\lambda(E, F)$. In the opposite direction we have the following result.

Theorem 7. If the Banach space $F$ has the extension property and if $T \in Q_\lambda(E, F)$ then $T \in N_\lambda(E, F)$ and $Q_\lambda(T) = N_\lambda(T)$.

We remark also that the above result is true for any pair $E$, $F$ provided the sequence space $\lambda$ is complemented. Thus for $\lambda = l^p$ when one gets the quasi-2-nuclear maps and the 2-nuclear maps, we have the (known) result that $N_2(E, F) = Q_2(E, F)$.

4. $\lambda$-nuclear maps and absolutely $\lambda$-summing maps. The linear map $T$ on $E$ into $F$ is said to be absolutely $\lambda$-summing if for each $x = (x_n) \in \lambda[E]$, the sequence $Tx = (Tx_n) \in \lambda[F]$. Let now $\lambda = \{x \in \omega : \rho(x) < \infty \}$.

Theorem 8. The linear map $T$ on $E$ into $F$ is absolutely $\lambda$-summing if and only if there exists a $\rho > 0$ such that for each finite system of vectors $x_1, x_2, \ldots, x_k$ in $E$ the following inequality holds:

$$\| (Tx_1, Tx_2, \ldots, Tx_k, 0, 0, \ldots) \|_\sigma \leq \rho \cdot \delta(x_1, x_2, \ldots, x_k, 0, 0, \ldots).$$

The smallest such $\rho$ is denoted $\pi_\lambda(T)$. It can be shown that when $F$ is a Banach space the space $\pi_\lambda(E, F)$ of all the absolutely $\lambda$-sum-
ming maps on $E$ into $F$ is a Banach space with the norm defined by $\pi_\lambda(\cdot)$.

The space $\lambda$ is said to have the norm iteration property if for each sequence $(x^n)$ of elements of $\lambda$ we have $p[p(x^n)] = p[p(x_i)]$ where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}, \ldots)$. It is easily verified that the spaces $c_0$ and $l^p$ ($1 \leq p \leq \infty$) have the above property.

**Theorem 9.** If $\lambda$ has the norm iteration property then $N_\lambda(E, F) \subset \pi_\lambda(E, F)$ and $\pi_\lambda(T) \subseteq N_\lambda(T)$.

We remark now that Theorem 9 above is true also for quasi-$\lambda$-nuclear maps with practically the same proof as that of Theorem 9. In case $\lambda = l^p$ ($p \geq 1$) the results of Persson and Pietsch [5] show that by taking the composition product of a certain finite number of $p$-quasi-nuclear maps one can obtain ultimately a nuclear map. In a rather general set up as ours we cannot prove a result of that type. Consequently when one attempts to formulate the concept of a $\lambda$-nuclear space using the standard canonical mappings, one obtains naturally two related concepts, those of $\lambda$-nuclear spaces and of quasi-$\lambda$-nuclear spaces.

**References**


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