LIE ALGEBRAS OF ANALYTIC VECTOR FIELDS AND UNIQUENESS IN THE CAUCHY PROBLEM FOR FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Let $P(x, D)$ be a partial differential operator defined in an open set \( \Omega \subset \mathbb{R}^n \) and let \( x^0 \in \Omega \) be a boundary point of a closed subset \( F \) of \( \Omega \). We say that there is uniqueness in the Cauchy problem (UCP) for the system \( (P, x^0, F) \) if to every open neighborhood \( U \subset \Omega \) of \( x^0 \) there is an open neighborhood \( V \subset U \) of \( x^0 \) such that for every distribution \( u \) in \( U \),

\[
P(x, D)u = 0 \quad \text{in } U, \quad \text{supp } u \subset F \cap U \quad \Rightarrow \quad u = 0 \quad \text{in } V.
\]

The classical uniqueness theorem of Holmgren (as extended to distribution solutions by Hörmander [1]) gives a sufficient condition for UCP for the system \( (P, x^0, F) \) in the case in which \( P \) is a linear partial differential operator with analytic coefficients and the boundary of \( F \) is a \( C^1 \) hypersurface \( S \). This condition is that \( S \) is not characteristic with respect to \( P \) at \( x^0 \). Although this condition is sufficient for UCP it is certainly not necessary. Malgrange [2], Hörmander [1], Trèves [3] and Zachmanoglou [4], [5], [6] have obtained some necessary and some sufficient conditions for UCP but the general problem is still unsolved.

In this note we present a necessary and sufficient condition for UCP for first order linear partial differential operators with analytic complex valued coefficients. No additional assumptions on the closed set \( F \) are made.

Let \( \mathcal{A} \) denote the ring of all real-valued analytic functions in \( \Omega \) and let

\[
(1) \quad P(x, D) = A + iB + c(x) = \sum_{j=1}^{n} a^j(x) D_j + i \sum_{j=1}^{n} b^j(x) D_j + c(x),
\]

where \( a^1, \ldots, a^n, b^1, \ldots, b^n \), Re \( c \) and Im \( c \) belong to \( \mathcal{A} \), \( i = \sqrt{-1} \) and \( D_j = \partial / \partial x_j \). \( A \) and \( B \) can be thought of as vector fields with coefficients in \( \mathcal{A} \). A trajectory of a collection \( \mathcal{C} \) of analytic vector fields is

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a piecewise analytic curve each analytic piece of which is an integral curve of an element of $\mathcal{C}$. Let

$$\mathcal{A}(A, B) = \{ \alpha A + \beta B : \alpha, \beta \in \mathcal{A} \}.$$ 

The following theorem asserts that the zeroes of solutions of $P(x, D)u = 0$ propagate along trajectories of $\mathcal{A}(A, B)$.

**Theorem 1.** For any distribution $u$ in $\Omega$ and any open subset $\Omega_0$ of $\Omega$,

$$P(x, D)u = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{in } \Omega_0 \Rightarrow u = 0 \quad \text{in } \tilde{\Omega}_0(P, \Omega)$$

where $\tilde{\Omega}_0(P, \Omega)$ is the set of points of $\Omega$ which can be connected to points of $\Omega_0$ by trajectories of $\mathcal{A}(A, B)$ contained in $\Omega$.

The proof is based on a general theorem concerning the propagation of zeroes of solutions of linear partial differential equations with flat characteristic cones [7].

Let $\mathcal{W}(\mathcal{A}(A, B))$ denote the set of points in $\Omega$ which can be connected to $x^0$ by trajectories of $\mathcal{A}(A, B)$ contained in $\Omega$. It is easy to see that Theorem 1 implies that the following condition is sufficient for UCP for the system $(P, x^0, F): \mathcal{W}(\mathcal{A}(A, B))$ intersects the complement of the set $F$ in every neighborhood of $x^0$. Thus it becomes necessary to study closely the set $\mathcal{W}(\mathcal{A}(A, B))$, at least in some neighborhood of the point $x^0$. It turns out that the nature of this set depends on the Lie algebra generated by the vector fields $A$ and $B$.

The bracket of two analytic vector fields is defined by $[A, B] = AB - BA$ and it is also an analytic vector field. The bracket operation has certain well-known properties which will not be mentioned here. The Lie algebra generated by $A$ and $B$ is denoted by $\mathcal{L}(A, B)$ and is defined as the set of all linear combinations with coefficients in $\mathcal{A}$ of $A, B$ and all vector fields obtained by repeated application of the bracket operation on $A$ and $B$. By $\dim \mathcal{L}(A, B)|_{x^0}$ we denote the dimension of the vector space obtained from $\mathcal{L}(A, B)$ by evaluating the coefficients at $x^0$. Clearly $\dim \mathcal{L}(A, B)|_{x^0}$ may vary from point to point in $\Omega$ but we always have

$$0 \leq \dim \mathcal{L}(A, B)|_{x^0} \leq n.$$ 

The analyticity of the vector fields $A$ and $B$ implies the following interesting theorem.

**Theorem 2.** Let $x^0$ be any point in $\Omega$ and suppose that

$$\dim \mathcal{L}(A, B)|_{x^0} = k,$$

where $0 \leq k \leq n$. Then there is an open neighborhood $U \subset \Omega$ of $x^0$ and a
A $k$-dimensional manifold $\mathfrak{M}^g(A, B)$ passing through $x^0$ and contained in $U$ and such that at every point of $\mathfrak{M}^g(A, B)$,

(i) $\dim \mathfrak{L}(A, B) = k$,

(ii) every element of $\mathfrak{L}(A, B)$ is interior (tangent) to $\mathfrak{M}^g(A, B)$.

When $k = 0$ or $k = n$ the conclusions of the theorem are immediate and do not depend on the analyticity of $A$ and $B$. However when $1 \leq k < n$ the assumption of analyticity is essential. When $k \geq 1$ we may assume that $A\big|_{x^0} \neq 0$ and the theorem is proved by showing that there is an analytic transformation of coordinates such that in the new coordinates, having origin corresponding to $x^0$ and in some neighborhood of the origin, $A$ and $B$ have the form,

$$A = D_1, \quad B = B^{(k)} + B^{(l)},$$

(3)

$$B^{(k)}(x, D^{(k)}) = b^1(x)D_1 + \cdots + b^k(x)D_k,$$

$$B^{(l)}(x, D^{(l)}) = b^{k+1}(x)D_{k+1} + \cdots + b^n(x)D_n,$$

where

(4)

$$B^{(k)}(x, D^{(k)}) \big|_{x^0} = 0,$$

(5)

$$\dim \mathfrak{L}(A, B^{(k)}) \big|_{x^0} = k$$

and, moreover,

(6)

$$\mathfrak{L}(A, B) \big|_{x^0} = \mathfrak{L}(A, B^{(k)}) \big|_{x^0}.$$

Here we use the notation $x^{(k)} = (x_1, \ldots, x_k), D^{(k)} = (D_1, \ldots, D_k), x^{(l)} = (x_{k+1}, \ldots, x_n), D^{(l)} = (D_{k+1}, \ldots, D_n)$. Note that the equation $x^{(k)} = 0$ defines the $k$-dimensional manifold $\mathfrak{M}^g(A, B)$.

At the time of the typing of this announcement it was brought to the attention of the author that Theorem 2 is a special case of a general theorem on Lie algebras of analytic vector fields on an analytic manifold, published in 1966 by Nagano [8].

Let $\mathfrak{M}^g(\mathfrak{L}(A, B))$ denote the set of points in $\Omega$ which can be connected to $x^0$ by trajectories of $\mathfrak{L}(A, B)$ contained in $\Omega$. In view of Theorem 2 the following theorem is immediate and it provides a means for constructing the manifold $\mathfrak{M}^g(A, B)$ by solving ordinary differential equations.

**Theorem 3.** In some neighborhood of $x^0$,

$$\mathfrak{M}^g(A, B) = \mathfrak{M}^g(\mathfrak{L}(A, B)).$$

The following theorem leads us back to our original problem of uniqueness in the Cauchy problem.
Theorem 4. In some neighborhood of $x^0$,

$$\mathfrak{M}^{x^0}(A, B) = \mathfrak{M}^x(\mathfrak{a}(A, B)).$$

In view of Theorem 2 it is enough to show that if $\dim \mathfrak{L}(A, B)|_{x=x^0} = n$ then every point in some neighborhood of $x^0$ can be connected to $x^0$ by a trajectory of $\mathfrak{a}(A, B)$ contained in that neighborhood.

Let us denote the manifold described in Theorems 2, 3 and 4 by $\mathfrak{M}(P, x^0)$. In view of Theorems 1 and 4 we will call $\mathfrak{M}(P, x^0)$ the zero propagator of $P(x, D)$ at $x^0$,

$$\mathfrak{M}(P, x^0) = \mathfrak{M}^{x^0}(A, B) = \mathfrak{M}^{x^0}(\mathfrak{a}(A, B)) = \mathfrak{M}^x(\mathfrak{a}(A, B)).$$

In the language of differential geometry, $\mathfrak{M}(P, x^0)$ is the maximal integral manifold passing through $x^0$ of the Lie subalgebra of analytic vector fields on $\Omega$ generated by the real and imaginary parts of the principal part of $P(x, D)$. Now, combining Theorems 1 and 4 we obtain a sufficient condition for UCP.

Theorem 5. Let $\Omega$ be an open set in $\mathbb{R}^n$, $P(x, D)$ a linear first order partial differential operator with analytic complex-valued coefficients in $\Omega$ and $x^0 \in \Omega$ a boundary point of a closed subset $F$ of $\Omega$. There is uniqueness in the Cauchy problem for the system $(P, x^0, F)$ if for every open neighborhood $U \subset \Omega$ of $x^0$,

$$\mathfrak{M}(P, x^0) \cap (U \sim F) \neq \emptyset,$$

i.e. the zero propagator of $P(x, D)$ at $x^0$ intersects the complement of $F$ in every neighborhood of $x^0$.

Corollary. If $\dim \mathfrak{L}(A, B)|_{x=x^0} = n$ then there is always uniqueness in the Cauchy problem for the system $(P, x^0, F)$ for any closed set $F$.

Thus, if at each point of an open set $\Omega \subset \mathbb{R}^n$, $\dim \mathfrak{L}(A, B) = n$ then the zeroes of solutions of the equation $P(x, D)u = 0$ propagate in exactly the same way as those of elliptic equations: For any open subset $\Omega_0$ of $\Omega$ and any distribution $u$ in $\Omega$, the conditions $P(x, D)u = 0$ in $\Omega$ and $u = 0$ in $\Omega_0$ imply that $u = 0$ in every connected component of $\Omega$ which intersects $\Omega_0$.

Theorem 6. If the principal part of $P(x, D)$ does not vanish at $x^0$ then condition (7) is also necessary for uniqueness in the Cauchy problem for the system $(P, x^0, F)$.

Theorem 6 is proved using formulas (3) and (4) and showing that
there is a solution $u$ of $P(x, D)u = 0$ in some open neighborhood of $x^0$ such that $\text{supp } u = \mathfrak{M}(P, x^0)$.

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