ON THE AVERAGE ORDER OF SOME ARITHMETICAL FUNCTIONS

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Abstract. We consider a large class of arithmetical functions generated by Dirichlet series satisfying a functional equation with gamma factors. Our objective is to state some $\Omega$ results for the average order of these arithmetical functions.

Our objective here is to state some $\Omega$-theorems on the average order of a class of arithmetical functions.

We indicate very briefly the class of arithmetical functions under consideration. For a more complete description, see [4].

Let $\{a(n)\}$ and $\{b(n)\}$ be two sequences of complex numbers, not identically zero. Let $\{X_n\}$ and $\{f_n\}$ be two strictly increasing sequences of positive numbers tending to $\infty$. Put $s = \sigma + it$ with $\sigma$ and $t$ both real and suppose that

$$\phi(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \quad \text{and} \quad \psi(s) = \sum_{n=1}^{\infty} b(n)n^{-s}$$

each converge in some half-plane. Let $\sigma^*$ denote the abscissa of absolute convergence of $\psi$. Put

$$\Delta(s) = \prod_{r=1}^{N} \Gamma(\alpha_r s + \beta_r),$$

where $\alpha_r > 0$ and $\beta_r$ is complex, $r = 1, \cdots, N$. Assume that for some real number $r$, $\phi$ and $\psi$ satisfy the functional equation $\Delta(s)\phi(s) = \Delta(r-s)\psi(r-s)$.

We shall consider the Riesz sum

$$A_q(x) = \frac{1}{\Gamma(q + 1)} \sum_{n \leq x} a(n)(x - \lambda_n)^q,$$

where $q \geq 0$. Let $\alpha = \sum_{r=1}^{N} \alpha_r$ and define

$$Q_q(x) = \frac{1}{2\pi i} \int \frac{\Gamma(s)\phi(s)x^{s+q}}{c_q \Gamma(s + q + 1)} \, ds,$$

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where \( C_q \) is a cycle encircling all of the singularities of the integrand to the right of \( \sigma = -q - 1 - k \), where \( k > |\frac{1}{2}r - 1/(4\alpha)| \), and where all of the singularities of \( \phi \) lie in \( \sigma > -k \). Then, the "error term" \( P_q(x) \) is defined by

\[
P_q(x) = A_q(x) - Q_q(x).
\]

Furthermore, let

\[
\beta(q) = \beta = - \sum_{r=1}^{N} \beta_r + \frac{1}{2} N - \frac{1}{2} r\alpha - \frac{1}{2} - \frac{1}{2} q,
\]

\[
\theta(q) = \theta = \frac{1}{2} r - 1/(4\alpha) + q - q/(2\alpha),
\]

and

\[
\kappa(q) = \kappa = \sigma^* - \frac{1}{2} r - 1/(4\alpha) - q/(2\alpha).
\]

From [4, p. 111], \( \kappa(0) \geq 0 \). In the sequel we assume that \( \kappa(q) \geq 0 \). If \( \kappa(q) < 0 \), the order of \( P_q(x) \) can be determined exactly [4, Theorem 3.2].

We are now ready to state

**Theorem 1.** Assume that \( b(n) \geq 0 \) and that \( \beta_r \) is real, \( r = 1, \ldots, N \). Suppose that there exist constants \( c \) and \( p \) such that as \( x \) tends to \( \infty \),

\[
\sum_{n} b(n) \sim c x^{\sigma^*} \log^{p-1} x.
\]

Lastly, suppose that \( \mu_{n+1} - \mu_n = o(\mu_n) \), as \( n \) tends to \( \infty \). Then, if \( \cos(\beta \pi) > 0 \) and \( \kappa > 0 \),

\[
\Re\{P_q(x)\} = \Omega_+(x^{\sigma} \{\log x\}^{\sigma} \{\log \log x\}^{\sigma-1});
\]

if \( \cos(\beta \pi) < 0 \) and \( \kappa > 0 \),

\[
\Re\{P_q(x)\} = \Omega_-(x^{\sigma} \{\log x\}^{\sigma} \{\log \log x\}^{\sigma-1}).
\]

The proof of Theorem 1 for \( q = 0 \) is given in [1]. The proof of the more general theorem given here follows along the same lines. The idea of the proof goes back to Szegö [7] and Szegö and Walfisz [8]. Dirichlet's theorem on the simultaneous approximation of a finite set of real numbers is used in the proof, and it is at this stage of the proof that the restriction \( b(n) \geq 0 \) is necessary.

Results of Hardy [5] on \( r_2(n) \), the number of representations of \( n \) as the sum of two squares, and on \( d(n) \), the divisor function, are special cases of Theorem 1. Results of Szegö [7] on \( r_2(n) \) and Szegö and Walfisz [8] on the Piltz divisor problem in algebraic number fields are also special cases.

For the arithmetical functions under consideration, Theorem 1 is an
improvement upon general theorems of Landau [6] and Chandrasekharan and Narasimhan [3], [4].

Theorem 1 yields only “one-sided” results. In many cases, however, we can obtain “two-sided” results as the following theorem shows.

**THEOREM 2.** Assume the hypotheses of Theorem 1. Let $Q_\psi(x)$ be $Q_\phi(x)$ except that $\phi$ is replaced by $\psi$. Suppose that as $x$ tends to $\infty$,

$$Q_\psi(x) \sim a_0 x^{2\gamma-1} \log^{\gamma-1} x.$$ 

Let $\gamma(q) = \gamma = 2\alpha \kappa - 1$, and for $\kappa > 0$ and a real define

$$g(a) = \int_0^\infty e^{-u^2} u^\gamma \cos(au + \beta \pi) du.$$ 

Then, if $\kappa > 0$ and $g(a)$ has a change in sign,

(1) \hspace{1cm} \text{Re}\{P_\phi(x)\} = \Omega_{\pm}(x^{\delta} \log x)^{\gamma};$$

if $\kappa = 0$, in all cases,

(2) \hspace{1cm} \text{Re}\{P_\phi(x)\} = \Omega_{\pm}(x^{\delta} \log x)^{\gamma}.$$

The assumption in Theorem 1 that $\cos(\beta \pi) \neq 0$ has been removed. However, we have an additional restriction in that $g(a)$ has a change in sign. In [2] we establish some general conditions under which $g(a)$ has a sign change. We also determine there some conditions under which $g(a)$ has no sign change. It is very unfortunate, indeed, that the most interesting cases of $r_q(n)$ and $d(n)$ for $q = 0$ fall into this latter category.

The proof of Theorem 2 for $q = 0$ and $k > 0$ is given in [2], and the proof for $q > 0$ follows along the same lines. For $\kappa = 0$, the proof is, in fact, somewhat easier. The idea for the proof of Theorem 2 goes back to Szegö and Walfisz [9], and so their results on the Piltz divisor problem for algebraic number fields are special cases of Theorem 2. Again, Dirichlet’s theorem is used in the proof, but in a different way, however.

Our next theorem yields some information on how often the inequalities (1) and (2) in Theorem 2 are valid.

**THEOREM 3.** Assume the hypotheses of Theorem 2. Then, there exist positive constants $c_1$ and $c_2$ and a positive, strictly increasing sequence $\{y_n\}$ tending to $\infty$ such that both inequalities

$$\pm \text{Re}\{P_\phi(x)\} > c_1 x^{\delta} (\log x)^{\gamma} (\log \log x)^{\gamma-1}$$
if $\kappa > 0$, and

$$\pm \Re \{ P_q(x) \} > c_1 x^\kappa (\log \log x)^\kappa$$

if $\kappa = 0$, have solutions in each interval

$$y_n \leq x \leq y_n + c_2 y_n^{-1/2\alpha} (\log y_n)^{1/2 - 1/(2\alpha)}.$$

For $\kappa > 0$ and $q = 0$ the proof is given in [2]. The proof of the more general Theorem 3 is exactly the same. Theorem 3 gives an improvement upon a general theorem of Landau [6] for the arithmetical functions under consideration.

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**References**


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