Let $K$ be an algebraically closed field of prime characteristic $p$.

By a classical Lie algebra over $K$ we shall understand a Lie algebra $\mathfrak{g}$ obtained from a complex simple Lie algebra $\mathfrak{g}_C$ by the well-known procedure of Chevalley: see [7] [1], for example. In this note we announce some results on the representation theory of $\mathfrak{g}$ over $K$; proofs will appear elsewhere. All modules considered will be finite-dimensional and restricted, unless otherwise specified.

0. Preliminaries. Denote by $\Sigma$ the root system of $\mathfrak{g}_C$ relative to a Cartan subalgebra, and let $\Pi = \{\alpha_1, \ldots, \alpha_I\}$ be a simple system. Fix a Chevalley basis $\{X\alpha, \alpha \in \Sigma; H_i, 1 \leq i \leq I\}$ of $\mathfrak{g}_C$; if $\mathfrak{g}_Z$ is the $\mathbb{Z}$-span of this basis, then $\mathfrak{g} = \mathfrak{g}_Z \otimes \mathbb{K}$. For convenience, we also denote by $X\alpha$, $H_i$ the corresponding elements of $\mathfrak{g}$. Write $\mathfrak{h} = \mathfrak{h}_Z \otimes \mathbb{K}$ (the $\mathbb{Z}$-span of the $H_i$ in $\mathfrak{g}$). Kostant’s theorem [7, §2] describes the $\mathbb{Z}$-form $\mathfrak{u}_Z$ of the universal enveloping algebra of $\mathfrak{g}_C$ generated by all $X\alpha/m!$ ($\alpha \in \Sigma$, $m \geq 0$).

If we let $V_\lambda$ be the irreducible $\mathfrak{g}_C$-module of highest weight $\lambda$, and let $v_0 \in V_\lambda$ be a maximal vector (a nonzero vector annihilated by all $X\alpha$, $\alpha \in \Pi$), then $\mathfrak{u}_Z v_0$ is an “admissible lattice.” Tensoring with $\mathbb{K}$ yields a (restricted) $\mathfrak{g}$-module $V_\lambda$, which is also a module for the simply connected Chevalley group $G$ constructed from $\mathfrak{g}_C$ over $K$. If $v_0$ again denotes the maximal vector $v_0 \otimes 1$ in $V_\lambda$, then $v_0$ has weight $\lambda$.

Let $\Lambda$ denote the collection of $p^i$ restricted weights $\lambda$ characterized by the conditions $0 \leq \lambda(H_i) < p$, $1 \leq i \leq l$. For each $\lambda \in \Lambda$ let $M_\lambda$ be the irreducible $\mathfrak{g}$-module of highest weight $\lambda$; it is known that $M_\lambda$ is a homomorphic, but not always isomorphic, image of $V_\lambda$. The collection $\mathfrak{M} = \{M_\lambda | \lambda \in \Lambda\}$ exhausts the (isomorphism classes of) irreducible $\mathfrak{g}$-modules. Let $\mathfrak{u}$, $\mathfrak{g}$ be the restricted universal enveloping algebras of $\mathfrak{g}$, $\mathfrak{h}$ over $K$ ($\mathbb{Z}$-algebras). (Left) $\mathfrak{u}$-modules correspond precisely to restricted (left) $\mathfrak{g}$-modules. Every $\mathfrak{u}$-algebra is a Frobenius algebra, and $\mathfrak{u}$ is even symmetric.
1. Standard cyclic modules and characters.

**DEFINITION.** A cyclic $g$-module, generated by a maximal vector (of weight $\lambda$), will be called standard cyclic (of weight $\lambda$).

**Proposition 1.** If $\lambda \in \Lambda$, the $g$-module $V_\lambda$ is standard cyclic of weight $\lambda$.

**Proposition 2 (Braden).** A standard cyclic $g$-module (restricted or not) is indecomposable and possesses a unique maximal submodule.

In characteristic 0 the “most general” standard cyclic module for $g_\mathbb{C}$ is always infinite-dimensional [6], [8], [9]. Here we consider the analogue for $g$. If $\{\beta_1, \ldots, \beta_m\}$ is the set of positive roots (relative to $\Pi$), let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_m$ be the corresponding $X_{\beta_i}$ and $X_{-\beta_i}$, respectively. Let $\mathfrak{h}$, $\mathfrak{n}'$ be the subalgebras of $g$ spanned by the $X_i$, $Y_i$ respectively, and let $\mathfrak{g}$, $\mathfrak{k}'$ be their $u$-algebras. If $\lambda \in \Lambda$, denote by $L_\lambda$ the left ideal in $\mathfrak{u}$ generated by all $X_i (1 \leq i \leq m)$ and all $H_i - \lambda(H_i) \cdot 1$ $(1 \leq i \leq l)$. Set $Z_\lambda = \mathfrak{u}/L_\lambda$. The canonical map $\mathfrak{u} \to Z_\lambda$ induces a vector space isomorphism of $\mathfrak{g}'$ onto $Z_\lambda$: indeed, the coset of 1 in $Z_\lambda$ is a maximal vector of weight $\lambda$, forcing $\dim Z_\lambda \geq p^m = \dim \mathfrak{g}'$, and on the other hand one can verify that $\mathfrak{g}' \cap L_\lambda = 0$. Moreover, any standard cyclic $g$-module of weight $\lambda$ is a homomorphic image of this “universal” one.

Next we introduce certain “characters” analogous to those of Harish-Chandra [6, Exposé 19]. Let $\mathfrak{c}$ be the center of $\mathfrak{u}$. Since $Z_\lambda$ is indecomposable (Proposition 2), Fitting’s Lemma allows one to show that each $C \in \mathfrak{c}$ acts as a scalar plus a nilpotent; in particular, the function $\chi_\lambda: \mathfrak{c} \to K$ assigning to $C$ its single eigenvalue on $Z_\lambda$, is a homomorphism of $K$-algebras. Moreover, $\chi_\lambda(C)$ is the single eigenvalue of $C$ on any subhomomorphic image of $Z_\lambda$, from which we deduce:

**Proposition 3.** $\chi_\lambda = \chi_\mu$ if $M_\lambda, M_\mu$ occur as composition factors of some standard cyclic $g$-module.

2. Linked weights and blocks.

**Definition.** Let $W$ be the Weyl group of $g_\mathbb{C}$, $\rho =$ half-sum of positive roots. If $\lambda, \mu \in \Lambda$, viewed as functions on $\mathfrak{h}$, satisfy: $\lambda + \rho = (\mu + \rho)^\sigma$ for some $\sigma \in W$, then we say $\lambda$ and $\mu$ are linked and write $\lambda \sim \mu$.

It is clear that linkage is an equivalence relation on $\Lambda$, since $(\lambda_\sigma)_\sigma = \lambda_{\tau_\sigma}$, where we write $\lambda_\sigma = (\lambda + \rho)^\sigma - \rho$. There is always a linkage class having only one member: take $\lambda = (\rho - 1)\rho$; this weight yields the “Steinberg module” $M_\lambda = V_\lambda = Z_\lambda$, the unique irreducible $g$-module of maximal dimension $p^m$. The condition $\lambda \sim \mu$ is analogous to Harish-
Chandra's condition for equality of "characters" in the infinite-dimensional case [6, Exposé 19].

**Theorem 1.** \( \lambda \sim \mu \) implies \( \chi_\lambda = \chi_\mu \).

Although a precise description of the submodules of \( Z_\lambda \) is lacking, the following can be shown.

**Proposition 4.** \( \lambda \sim \mu \) implies that \( Z_\lambda \) and \( Z_\mu \) have the same composition factors (multiplicities counted). Up to scalar multiples, \( Z_\lambda \) has a unique minimal vector, namely, the coset of \( Y_1^{-1} \cdots Y_m^{-1} \) (for any ordering of \( Y_1, \cdots, Y_m \)).

The linkage class of \( \lambda \) is in 1-1 correspondence with the \( W \)-orbit of \( \lambda + \rho \) in \( \Lambda \), so Theorem 1 shows there are no more characters than orbits. We can relate this to the blocks of \( \mathfrak{u} \) as well [4, §55]. The distinct (left) principal indecomposable modules (PIM's) of \( \mathfrak{u} \) correspond 1-1 with the elements of \( \mathfrak{A} \): The PIM \( U_\lambda \) has unique highest composition factor \( M_\lambda \). Two PIM's are said to be "linked" if they share a composition factor, and the sum of all PIM's in a class of this equivalence relation is an indecomposable two-sided ideal of \( \mathfrak{u} \), called a "block." Let \( B_\lambda \) be the block containing \( U_\lambda \). It is easy to see that (under the canonical map \( \mathfrak{u} \rightarrow Z_\lambda \)) some copy of \( U_\lambda \) maps onto \( Z_\lambda \), whence every composition factor of \( Z_\lambda \) belongs to the block \( B_\lambda \). In view of Theorem 1 and Proposition 4, we can state:

**Theorem 1'.** \( \lambda \sim \mu \) implies \( U_\lambda \) and \( U_\mu \) are linked (so \( B_\lambda = B_\mu \)).

This shows that the number \( t \) of distinct blocks does not exceed the number of \( W \)-orbits in \( \Lambda \) (and each block corresponds to a union of such orbits). Moreover, \( t = \dim(\mathfrak{C}/(\mathfrak{C} \cap \mathfrak{R})) \), and the \( \chi_\lambda \) coincide with the homomorphisms \( \mathfrak{C} \rightarrow K \) defined by the respective block idempotents [4, §85 and references].

3. **Invariants.** In order to prove the converse of Theorem 1 (under some restriction on \( \rho \)) it is necessary to examine more closely how \( \mathfrak{C} \) acts on \( Z_\lambda \). There is a natural \( K \)-linear map \( \beta: \mathfrak{U} \otimes \mathfrak{C} \otimes \mathfrak{C} \rightarrow \mathfrak{C} \) defined by \( \beta(YHX) = 0 \) if \( Y \) or \( X \) is not 1, \( \beta(YHX) = H \) if \( Y = X = 1 \) \( (Y \in \mathfrak{U}, H \in \mathfrak{C}, X \in \mathfrak{C} \) standard basis monomials). If \( \lambda \in \Lambda \) is viewed as a \( K \)-algebra homomorphism \( \mathfrak{C} \rightarrow K \), then in view of the way \( \chi_\lambda \) was defined, we have \( \chi_\lambda (C) = \lambda(\beta(C)) \), \( C \in \mathfrak{C} \), and moreover, \( \beta | \mathfrak{C} \) is multiplicative. Let \( \gamma \) be the \( K \)-algebra automorphism of \( \mathfrak{C} \) sending \( H_i \) to \( H_i - \rho(H_i) \) (\( \rho \) as before). Then Theorem 1 implies that \( \gamma(\beta(C)) \) lies in \( \mathfrak{C}^{W} \) (= algebra of \( W \)-invariants in \( \mathfrak{C} \)), so \( \dim \mathfrak{C}^{W} = t' \geq t \). Now \( \mathfrak{C}^{W} \) is a commutative semisimple associative algebra, and the corresponding \( t' \) \( K \)-algebra homomorphisms \( \mathfrak{C}^{W} \rightarrow K \) are just the restric-
tions to $\mathfrak{C}^W$ of the $\lambda \in \Lambda$, those which are $W$-conjugate having the same restriction (so $t' =$ number of $W$-orbits in $\Lambda$). To prove the converse of Theorem 1, it would suffice to prove that $t = t'$, or that $\gamma(\beta(C)) = 3\mathfrak{C}^W$. This seems likely to hold in general, but our method, based on reduction mod $p$, does not work for “small” $p$.

**Theorem 2.** If $p > \text{Coxeter number of } \Sigma$, then $\chi_\lambda = \chi_\mu$ implies $\lambda \sim \mu$.

**Remark.** The Coxeter number $h$ (=order of the product of all simple reflections in $W$) for each of the simple types is as follows [2, pp. 250–275]: $A_l$, $l+1$; $B_l$, $C_l$, $2l$; $D_l$, $2l-2$; $E_6$, 12; $E_7$, 18; $E_8$, 30; $F_4$, 12; $G_2$, 6. If $p > h$, $p$ does not divide the order of $W$.

4. **Projective modules.** We recall [4, §56] that the projective $\mathfrak{U}$-modules are just the direct sums of the PIM's (which are the only indecomposable projectives). It is easy to see that if $M$ is indecomposable and $P \rightarrow M \rightarrow 0$ is a projective cover, then a sum of PIM’s from the same block already maps onto $M$. Since every $\mathfrak{U}$-module has a projective cover, we deduce from Theorem 2:

**Theorem 3.** If $p > h$, then if $M$ is an indecomposable $\mathfrak{U}$-module, all composition factors of $M$ have highest weights which are linked.

This has been conjectured in general by Verma; Pollack’s study of type $A_l$ confirms the result directly [5], and Braden’s conclusions [3] are highly consistent with it.

In [5] Pollack describes the PIM’s for $A_l$ explicitly. For higher ranks we get some analogous results, the first of which resembles a classical theorem on group algebras of finite groups [4, 65.17].

**Proposition 5.** Every projective $\mathfrak{U}$-module is projective as $\mathfrak{U}'$-module; in particular, each PIM of $\mathfrak{U}$ has dimension divisible by $p^m$ ($m =$ number of positive roots).

**Proposition 6.** If $\mathfrak{B}'$ is the subalgebra of $\mathfrak{U}$ generated by $\mathfrak{C}$ and $\mathfrak{B}'$, then every projective $\mathfrak{U}$-module is a projective $\mathfrak{B}'$-module. The PIM’s of $\mathfrak{B}'$ are just the $p^t$ modules $Z_\lambda$ ($\lambda \in \Lambda$) regarded as $\mathfrak{B}'$-modules.

The proof of Proposition 6 is a direct construction in $\mathfrak{U}$. Using this result, along with Proposition 4, one can get precise information about dimensions.

**Theorem 4.** Let $C$ be the Cartan matrix of $\mathfrak{U}$ $(c_{\mu \nu} =$ multiplicity of $M_\mu$ as composition factor of $U_\nu$), and let $D$ be the matrix $(d_{\lambda \mu})$, where $d_{\lambda \mu} =$ multiplicity of $M_\mu$ as a composition factor of $Z_\lambda$. Whenever the
conclusion of Theorem 3 is valid, \( C = D \cdot D \), \( \dim U_\lambda = a_\lambda d^\lambda p^m \) and \( \dim B_\lambda = a_\lambda p^{2m} \), where \( a_\lambda \) = cardinality of \( W \)-orbit of \( \lambda + \rho \) in \( \Lambda \).

References

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