NONLINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS
AND THE GENERALIZED TOPOLOGICAL DEGREE

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Introduction. It is our purpose in the present note to present a
general existence theorem for noncoercive elliptic boundary value
problems for operators of the form:

\[ A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \cdots, D^m u), \]

on closed subspaces \( V \) of the Sobolev space \( W^{m,p}(G), G \) an open subset
of \( \mathbb{R}^n \), \( n \geq 1 \). This existence theorem is based upon an extension of the
theory of the generalized topological degree for \( A \)-proper mappings
of Banach spaces introduced in Browder-Petryshyn \([8],[9]\), and, in
particular, on an extension of the Borsuk-Ulam theorem to pseudo-
monotone mappings \( T \) from a reflexive separable Banach space \( V \)
to its conjugate space \( V^* \).

To make a precise statement of our general existence theorem
possible, we introduce the following notation: For a given \( m \geq 1 \), we
let \( \xi \) be the \( m \)-jet of a function \( u \) from \( \mathbb{R}^n \) to \( \mathbb{R}^s \) for some given \( s \geq 1 \),
i.e. \( \xi = \{ \xi_\alpha : |\alpha| = m \} \), and set

\[ \xi = \{ \xi_\alpha : |\alpha| = m \}, \quad \eta = \{ \eta_\beta : |\beta| \leq m - 1 \}, \]

where each \( \xi_\alpha, \xi_\beta, \) and \( \eta_\beta \) is an element of \( \mathbb{R}^s \). The set of all \( \xi \) of the
above form is an Euclidean space \( \mathbb{R}^{rm} \), and correspondingly, \( \xi \in \mathbb{R}^{rm} \),
\( \eta \in \mathbb{R}^{rms-1} \).

For each \( \alpha, A_\alpha \) is assumed to be a function from \( G \times \mathbb{R}^m \) to \( \mathbb{R}^s \) sati-
ifying the following conditions:

Assumptions on \( A(u) : (1) A_\alpha(x, \xi) \) is measurable in \( x \) for fixed \( \xi \) and
continuous in \( \xi \) for fixed \( x \). For a given \( \rho \) with \( 1 < \rho < \infty \), there exists a
constant \( c \) such that

\[ |A_\alpha(x, \xi)| \leq c \left( 1 + \sum_{|\beta| \leq m} |\xi_\beta|^{p_\beta} \right) \]

with \( p_\beta \leq (\rho - 1) \) for \( |\alpha| = |\beta| = m \), and

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limit of \( A \)-proper mappings.
\[ p_{\alpha} < \frac{n p + p(m - |\alpha|) - n}{n - p(m - |\beta|)} , \quad \text{if } m - \frac{n}{p} \leq |\alpha| \leq m, \]
\[ m - \frac{n}{p} \leq |\beta| \leq m, \]
\[ |\beta| + |\alpha| < 2m, \]
\[ p_{\alpha} \leq \frac{n p}{n - p(m - |\beta|)} , \quad \text{if } |\alpha| < m - \frac{n}{p}, \]
\[ m - \frac{n}{p} \leq |\beta| \leq m. \]

(2) If \( \xi = (\xi', \eta) \), then for each \( x \) in \( G \), \( \eta \) in \( \mathbb{R}^{n-1} \), \( \xi \) and \( \xi' \) in \( \mathbb{R}^n \) with \( \xi \neq \xi' \),
\[ \sum_{|\alpha| = m} \langle A_{\alpha}(x, \xi, \eta) - A_{\alpha}(x, \xi, \eta), \xi_{\alpha} - \xi'_{\alpha} \rangle > 0, \]
(where \( \langle \cdot, \cdot \rangle \) denotes the inner product in \( \mathbb{R}^s \)).

(3) For each \( \gamma \) and \( \gamma' \) in \( \mathbb{R}^n \),
\[ \sum_{|\alpha| = m} \langle A_{\alpha}(x, \xi, \eta) - \gamma_{\alpha}, \xi_{\alpha} - \gamma'_{\alpha} \rangle \to \infty \quad (|\xi| \to \infty), \]
uniformly for bounded \( \eta \).

Let \( W^{m,p}(G) \) be the Sobolev space of \( s \)-vector functions \( u \) such that \( u \) and all its derivatives \( D^\alpha u \) for \( |\alpha| \leq m \) lie in \( L^p(G) \) where \( p \) is the exponent involved in the inequalities of Assumption (1). Then for any \( u \) and \( v \) in \( W^{m,p}(G) \), we may define the generalized Dirichlet form corresponding to the representation (1) by:

(2) \[ a(u, v) = \sum_{|\alpha| \leq m} (A_{\alpha}(\xi(u)), D^\alpha v), \]
where
\[ \xi(u) = \{ D^\alpha u : |\alpha| \leq m \}, \quad A_{\alpha}(\xi(u))(x) = A_{\alpha}(x, \xi(x)(x)), \]
\[ (w, v) = \int_G \langle w(x), u(x) \rangle dx, \quad \text{(integration with respect to Lebesgue } n\text{-measure).} \]

**Theorem 1.** Let \( G \) be an open subset of \( \mathbb{R}^n \) with \( G \) bounded and the Sobolev Imbedding Theorem valid on \( G \) (i.e. \( G \) satisfies mild smoothness conditions on its boundary). Let \( A(u) \) be a quasilinear elliptic operator of order \( 2m \) on \( G \) of the form

(1) \[ A(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_{\alpha}(\xi(u)), \]
where the coefficient functions $A_\alpha$ satisfy Assumptions (1), (2), and (3) above. Suppose that $A(u)$ is odd in $u$, i.e. $A_\alpha(x, -\xi) = -A_\alpha(x, \xi)$ for each $\alpha$ and all $x$ in $G$, $\xi$ in $\mathbb{R}^m$. For each $w$ in $V^*$, the dual space of a closed subspace $V$ of $W^{m,p}(G)$, consider the problem of finding $u$ in $V$ such that $a(u, v) = (w, v)$ for all $v$ in $V$. Suppose that there exists a continuous function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that for each solution $u$ of this problem for any $w$ in $V^*$,

\[
\|u\|_V = \|u\|_{W^{m,p}(G)} \leq \phi(\|w\|_{V^*}).
\]

Then for each $w$ in $V^*$, there exists at least one solution $u$ in $V$ of the problem: $a(u, v) = (w, v)$ for all $v$ in $V$.

We have used the notation $(w, v)$ in Theorem 1 to denote the pairing between $w$ in $V^*$ and $u$ in $V$.

**Theorem 2.** Let $G$ be a bounded, smoothly bounded open set in $\mathbb{R}^n$ (as in Theorem 1), and consider a one-parameter family of operators $A_t(u), t \in [0, 1]$, where for each $t$,

\[
A_t(u) = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(\xi(u); t)
\]

and the coefficient functions are continuous in $t$, uniformly for bounded $\xi$ and all $x$ outside a null set in $G$. For each $t$, we take the generalized Dirichlet form

\[
a_t(u, v) = \sum_{|\alpha| \leq m} (A_\alpha(\xi(u); t), D^\alpha v),
\]

where we assume that $A_t(u)$ satisfies Assumptions (1), (2), (3) for each $t$ in $[0, 1]$. Suppose that $A_t(u)$ is odd, and that there exists a continuous function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ such that if $a_t(u, v) = (w, v)$ for some $w$ in $V^*$, $u$ in $V$, $t$ in $[0, 1]$ and all $v$ in $V$, then $\|u\|_V \leq \phi(\|w\|_{V^*})$.

Then the problem: $a_0(u, v) = (w, v)$ for all $v$ in $V$; has a solution $u$ in $V$ for each $w$ in $V^*$.

Theorem 2 includes Theorem 1 as the special case in which $A_t(u) = A(u)$ for all $t$ in $[0, 1]$. It also includes the standard existence theorem for $A(u)$ in which the Dirichlet form $a(u, v)$ is assumed to be coercive, i.e.

\[
(6) \text{ There exists } c: \mathbb{R}^+ \to \mathbb{R}^1 \text{ with } c(r) \to \infty \text{ as } r \to \infty \text{ such that } a(u, u) \geq c(\|u\|_V) \|u\|_V.
\]

Indeed, if $A(u)$ is coercive, and if we set $A_t(u) = A(u) - tA(-u)$ for $t$ in $[0, 1]$, then $A_0(u) = A(u)$, $A_t(u)$ is odd, the Assumptions (1), (2), and (3) hold for every $A_t(u)$, while since $a_t(u, u) = a(u, u) - ta(-u, u) = a(u, u) + ta(-u, -u)$, it follows that
providing that \( \|u\|_V > R \), where \( c(r) > 0 \) for \( r > R \). Suppose that for some \( u \) in \( V \), \( w \) in \( V^* \) and \( t \) in \([0, 1]\), we have
\[
a_t(u, v) = (w, v) \quad (v \in V).
\]
Then:
\[
c(\|u\|_V) \|u\|_V \leq a_t(u, u) = (w, u) \leq \|w\|_{V^*} \|u\|_V,
\]
and as a consequence \( c(\|u\|_V) \leq \|w\|_{V^*} \) if \( u = 0 \). If we set \( \phi(s) = \sup \{r : c(r) \leq s\} \), it follows that \( \|u\|_V \leq \phi(\|w\|_{V^*}) \) and by Theorem 2, the equation \( a(u, v) = (w, v) \) \((v \in V)\), has a solution \( u \) in \( V \) for each \( w \) in \( V^* \).

Existence theorems for elliptic boundary problems of this type were first obtained by Visik [15] using compactness arguments and a priori estimates on \((m+1)\text{st}\) derivatives. Monotonicity arguments were first applied to these problems in Browder [2], [3], using the basic existence theorem for monotone maps from a reflexive Banach space \( V \) to \( V^* \) proved independently by Browder [2] and Minty [12]. The existence theorem in the coercive case was extended to elliptic operators \( A(u) \) satisfying Assumptions (1), (2), and (3) by Leray-Lions [11]. Borsuk-Ulam theorems for monotone and semimonotone operators in infinite dimensional Banach spaces were first derived by Browder [4], [5], and were first applied to odd, homogeneous, elliptic operators satisfying strong monotonicity conditions by Pohozaev [14]. Theorem 1 was first obtained under a stronger hypothesis \((3)'\) rather than (3) in Browder [6], together with Assumptions (1) and (2) on \( A(u) \). This is as follows:

\((3)'\)

There exist continuous functions \( k(\eta), k_0(\eta) > 0 \) such that
\[
\sum_{|\alpha| \leq m} \langle A_\alpha(x, \eta, \xi_a) \rangle \geq k_0(\eta) |\xi|^p - k(\eta),
\]
for all \( x \) in \( G \), \( \xi \) in \( \mathbb{R}^m \), \( \eta \) in \( \mathbb{R}^{m-1} \).

1. Proofs of Theorems 1 and 2 rest upon general results concerning two classes of nonlinear mappings of monotone type from a reflexive Banach space \( V \) to its conjugate space \( V^* \).

Definition 1. Let \( V \) be a Banach space, \( V^* \) its conjugate space, \( T \) a mapping from \( V \) to \( V^* \), Then:

(a) \( T \) is said to be pseudomonotone if for any sequence \( \{u_j\} \) in \( V \) with \( u_j \) converging weakly to \( u \) in \( V \) such that \( \sup(Tu_j, u_j - u) \leq 0 \), it follows that for any \( v \) in \( V \), \( \liminf (Tu_j, u_j - v) \geq (Tu, u - v) \).

(b) \( T \) is said to satisfy condition \((S)\) if for any sequence \( u_j \) in \( V \) with
\( \{u_j\} \) converging weakly to \( u \) in \( V \) for which \( \lim(Tu_j, u_j - u) \leq 0 \), it follows that \( u_j \) converges strongly to \( u \) in \( V \).

**Proposition 1.** Suppose that \( A \) satisfies Assumption (1). Then there exists a continuous bounded mapping \( T \) of \( V \) into \( V^* \) for a given closed subspace \( V \) of \( W^{m,p}(G) \) such that for all \( u \) and \( v \) of \( V \), \( (Tu, v) = a(u, v) \). If \( A(u) \) satisfies Assumptions (2) and (3), \( T \) is pseudomonotone. If \( A(u) \) satisfies Assumptions (2) and (3)', then \( T \) satisfies condition \((S)_+\).

The proof of Proposition 1 is given in §1 of [7], and Appendix to §1. Pseudomonotonicity was first defined by Brézis in [1] (though our definition differs slightly from his in considering only sequences rather than filters). The condition \((S)_+\) was first defined in connection with the study of nonlinear eigenvalue problems in Browder [6] and is studied in detail in Browder [7], [8].

**Theorem 3.** Let \( V \) be a reflexive separable Banach space, \( T \) a mapping of \( V \) into \( V^* \) which is pseudomonotone, bounded on bounded sets, and continuous from each finite dimensional subspace of \( V \) to the weak topology of \( V^* \). Then:

(a) If \( T \) is an odd mapping outside of some ball around the origin and if \( T^{-1}(B) \) is bounded for any bounded subset \( B \) of \( V^* \), then \( R(T) \), the range of \( T \), is all of \( V^* \).

(b) If \( \{T_t\} \) is a family of bounded, pseudomonotone, finitely continuous mappings from \( V \) to \( V^* \) which is continuous in \( t \) uniformly on bounded subsets of \( V \), with \( T_0 = T \), \( T_t \) odd outside some ball, and if there exists a function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) such that \( T_t(u) = w \) implies that

\[
\|w\| \leq \phi(\|w\|) \quad (t \in [0, 1]),
\]

then \( R(T) = V^* \).

Theorem 3 and Proposition 1 together imply the validity of Theorems 1 and 2. Theorem 3 follows from an extension to the class of pseudomonotone mappings from \( V \) to \( V^* \) of the theory of the generalized degree defined for \( A \)-proper mappings of Banach spaces in Browder-Petryshyn [9], [10] and applied to mappings \( T \) from a reflexive \( V \) to \( V^* \) satisfying condition \((S)\) in Chapter 17 of Browder [8]. The basic facts are summarized in the following theorem:

**Theorem 4.** Let \( V \) be a reflexive separable Banach space, \( V^* \) its conjugate space. Let \( T \) be a mapping from \( V \) to \( V^* \) which is finitely continuous from \( V \) to \( V^* \) (i.e. continuous from each finite dimensional subspace of \( V \) to the weak topology of \( V^* \)) and bounded (i.e. maps bounded subsets of \( V \) into bounded subsets of \( V^* \)). Then:
(a) If $T$ is pseudomonotone, there exists a sequence $\{T_j\}$ of finitely continuous, bounded mappings, each satisfying condition $(S)_+$, which converges to $T$ uniformly on every bounded subset of $V$.

(b) If $T$ satisfies condition $(S)_+$, then $T$ is $A$-proper in the following sense [9], [10]: If $B$ is a closed ball of $V$, $\{V_j\}$ an increasing sequence of finite dimensional subspaces of $V$ whose union is dense in $V$, and if for each $j$, $u_j$ is an element of $V_j \cap B$ such that for a given element $w$ of $V^*$,

$$\|\phi_j^* T u_j - \phi_j^* w\|_{V_j^*} \to 0 \quad (j \to \infty),$$

where $\phi_j$ is the injection map of $V_j$ into $V$, $\phi_j^*$ the projection map of $V^*$ onto $V_j^*$, then there exists an infinite subsequence $\{u_{j(k)}\}$ converging strongly to an element $u$ of $B$ such that $T(u) = w$.

The proof of Theorem 4 is given in Chapter 17 of Browder [8]. The second property tells us that the generalized degree theory of Browder-Petryshyn [10] applies to mappings $T$ satisfying the condition $(S)_+$ (for the details of this application, see [8]). The corresponding generalized degree theory for pseudomonotone maps follows from the convexity of the class of $T$ satisfying $(S)_+$ and the following theorem whose proof will be published elsewhere:

Theorem 5. Let $X$ and $Y$ be Banach spaces, $G$ a bounded open subset of $X$, and consider an oriented approximation scheme $\{(X_n, Y_n, P_n, Q_n)\}$ for mappings $T$ of $\text{cl}(G)$ into $Y$ in the sense of [10]. Let $Z$ be a convex family of $A$-proper mappings from $\text{cl}(G)$ to $Y$ with respect to the given approximation scheme. Let $T$ be a mapping from $\text{cl}(G)$ to $Y$ which is the uniform limit on $\text{cl}(G)$ of mappings $T_j$ from the class $Z$. Then:

(a) For any sequence $\{T_j\}$ from $Z$ converging to $T$, if $w$ does not lie in $\text{cl}(T(\text{bdry}(G)))$, then $\text{Deg}(T_j, G, w)$ is the same for all $j$ sufficiently large and does not depend upon the choice of $\{T_j\}$. We denote this limit as $\text{Deg}(T, G, w)$.

(b) $\text{Deg}(T, G, w)$ is invariant under homotopy and weakly additive in the sense of Theorem 1 of [10]. If $\text{Deg}(T, G, w) \neq \{0\}$ and if $T(\text{cl}(G))$ is closed in $Y$, then $w$ lies in $T(\text{cl}(G))$.

(c) If $T$ is odd in the sense of Theorem 1 of [10], then $\text{Deg}(T, G, 0)$ consists only of odd integers, and $\text{Deg}(T, G, 0) \neq \{0\}$.

Added in proof. Results closely related to Theorem 5 have also been obtained by P. M. Fitzpatrick in connection with his Rutgers Ph.D. dissertation.
Bibliography


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