THE INEQUALITY OF SQPS AND QSP AS OPERATORS ON CLASSES OF GROUPS

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Some years ago Evelyn Nelson asked, as a special case of a question of wider interest in universal algebra, whether if $\mathfrak{X}$ is a class of groups it is always the case that $\mathrm{SQPS} \mathfrak{X} = \mathrm{QSP} \mathfrak{X}$ (see [10], [11, Problem 3], [2] and [4, p. 161]). Here $\mathfrak{X}$ is the class of groups isomorphic to subgroups of groups in $\mathfrak{X}$; $\mathfrak{Q} \mathfrak{X}$ is the class of groups isomorphic to factor groups of groups in $\mathfrak{X}$; and $\mathfrak{P} \mathfrak{X}$ is the class of groups isomorphic to cartesian products of families of groups in $\mathfrak{X}$. My aim is to indicate a proof that, if $\mathrm{SL}(2, q)$ is the group of $2 \times 2$ matrices of determinant 1 with entries from the field $GF(q)$ of $q$ elements, and $\mathfrak{X} = \{ G | G \cong \mathrm{SL}(2, 2^m), m \geq 2 \}$, then $\mathrm{SQPS} \mathfrak{X} \neq \mathrm{QSP} \mathfrak{X}$.

The proof uses two special properties of the groups $\mathrm{SL}(2, 2^m)$.

**Fact 1** (cf. [3, Chapter 12], or [7, Kap.II, §8]). If $X$ is a subgroup of $\mathrm{SL}(2, 2^m)$ then either $X \cong \mathrm{SL}(2, 2^l)$ for some divisor $l$ of $m$, or $X \in \mathfrak{M}$, the class of metabelian groups. In fact, if $X$ is not of the form $\mathrm{SL}(2, 2^l)$, then one knows that $X$ is cyclic, or dihedral or a subgroup of the 1-dimensional affine group over $GF(2^m)$, but all we shall need is that such groups are metabelian.

**Fact 2.** If $m \geq 2$ then $\mathrm{SL}(2, 2^m)$ is simple. Moreover, there is an integer $k$ such that for all $m$ and all $g, h \in \mathrm{SL}(2, 2^m)$ with $g \neq 1$, $h$ can be written as a product of exactly $k$ conjugates of $g$. Here $k$ may be taken to be 12, and the proof is a straightforward calculation.

A crucial consequence of Fact 2 is that if $X = \prod_i S_i$, where $S_i \in \mathfrak{X}$ for all $i \in I$, and if $g \in X$ then the normal closure of $g$ is given by

$$\langle g \rangle^X = \prod_i \{ S_i | i \in \supp(g) \} \leq X,$$

where $\supp(g) = \{ i \in I | g(i) \neq 1 \}$. It follows easily that if $N \triangleleft X$ and $\mathfrak{X} = \{ E \subseteq I | E = \supp(g) \text{ for some } g \in N \}$, then $\mathfrak{E}$ is an ideal (cf. [4]) in the boolean algebra of subsets of $I$ and $N = N_\mathfrak{E} = \{ g \in X | \supp(g) \in \mathfrak{E} \}$. Therefore $X/N$ is a reduced product (cf. [4, p. 144], or [1, p. 210]) of the groups $S_i$. Also

$$N = N_\mathfrak{E} = \cap \{ N_M | M \subseteq \mathfrak{M} \text{ and } \mathfrak{M} \text{ a maximal ideal on } I \}.$$

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Thus, since $X/N_{\mathfrak{X}}$ is an ultraproduct ([1, p. 210], [4, p. 145]) of the groups $S_i$, we see that $X/N$ is residually an ultraproduct of the groups in $\mathfrak{X}$.

Now, from Fact 1, $s \mathfrak{X} \subseteq \mathfrak{X} \cup \mathfrak{M}$ and so, if $G \in \mathfrak{P}$ then $G = X \times Y$ where $X$ is a cartesian product of groups in $\mathfrak{X}$, and $Y$ is metabelian. If $N < G$ then it follows from Fact 2 that $N = (X \cap N) \times (Y \cap N)$ and $G/N \cong (X/X \cap N) \times (Y/Y \cap N)$. Thus if $H \in \mathfrak{Q}$ then $H = X \times Y$, where $X$ is residually an ultraproduct of groups in $\mathfrak{X}$ and $Y$ is metabelian. Since $\mathfrak{X}$ consists of 2-dimensional linear groups, an ultraproduct of groups in $\mathfrak{X}$ is a 2-dimensional linear group over a suitable field (see, for example, Kegel [8]). Furthermore, a linear group is locally residually finite (Mal'cev [9, Theorems VII, VIII]). Therefore the direct factor $X$ of $H$ is residually locally residually finite, that is $X$ is locally residually finite. Since, by a theorem of P. Hall [5], $Y$ is also locally residually finite, it follows that $H$, and then every subgroup of $H$ has this property. That is, $\mathfrak{Q} \cong \mathfrak{X}$ consists of locally residually finite groups.

On the other hand, $\mathfrak{Q} \cong \mathfrak{X}$ is the variety generated by $\mathfrak{X}$, and this is known [6, p. 45] to be the variety $\mathfrak{D}$ of all groups. Since there is no dearth of groups, such as the finitely generated infinite simple groups, or finitely generated non-Hopf groups, or the group of permutations of the integers generated by the infinite cycle $\cdots, -1, 0, 1, 2, 3, \cdots$ and the 3-cycle $(1, 2, 3)$, which are not locally residually finite, it follows that $\mathfrak{Q} \cong \mathfrak{X}$, as promised.

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References


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