K-THEORETIC INTERPRETATION OF TAME SYMBOLS ON $k(t)$

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In [3] we introduced a canonical resolution for computing the $K$-theory of [4] and we found a map $\psi: K_2(A) \to \kappa_2^{GL}(A)$ where $K_2(A)$ is the group defined by Milnor [5] and $\kappa_2^{GL}(A)$ is the group of [3]. The map $\psi$ was proved surjective if $A$ is a regular ring. In this announcement we indicated how to compute $\kappa_L^2(k(t))$ for the field $k(t)$ of rational functions in one variable $t$. As a byproduct of this work we have proved

THEOREM 1. Write $K_2(A[t, t^{-1}]) = K_2(A) \oplus X$. Then if $A$ is regular, $X$ has a homomorphic image $K_1(A)$.

I should like to thank H. Bass for suggesting that Theorem 1, which was buried in my original announcement, be set off as a main result. Bass has informed me that J. Wagoner also has results on the group $X$.

1. Generalities. If $R$ is any ring (without unit) recall the path ring $\Omega R = x(1-x)R[x]$. Clearly $\Omega(R[T]) = (\Omega R)[T]$ if $T$ is a free abelian group or monoid. Also $\kappa_2^{GL}(R) \cong \kappa_1^{GL}(\Omega R)$ [3].

Proposition 1. $\kappa_1^{GL}(R[t]) = \kappa_1^{GL}(R)$ and $\kappa_1^{GL}(R[t, t^{-1}]) = \kappa_1^{GL}(R) \oplus K_0(R^+)$. This is an easy consequence of results of [1] and [3].

Proposition 2. If $A$ is regular, then the composition $\omega$ is a surjective homomorphism.

Theorem 1 follows from this proposition using results of [1] and [5].


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1 ADDED IN PROOF. For the second conclusion of Proposition 1 we require that $R$ be of the form $\Omega^n S$ where $S$ is regular.
Let $R[[t]]$ be the ring of formal power series in $t$ and let $R((t)) = S^{-1}R[[t]]$ where $S = \{1, t, t^2, \cdots \}$.

**Proposition 3.** There is a split epimorphism $K_1(R((t))) \to K_0(R)$.

The map is that defined on p. 74 of [1]. We prove that the module $M(\delta)$ defined there is finitely generated and projective over $R$. The proof of this fact is however necessarily different from that offered in [1].

**Corollary.** The epimorphism of Proposition 3 factors through the quotient $K_1^\text{GL}(R((t)))$. Thus there is a commutative diagram

$$
\begin{array}{ccc}
K_2(A((t))) & \to & K_1^\text{GL}(A((t))) \\
\downarrow \psi & & \downarrow \omega \\
K_0((\Omega A)^+) & \leftarrow & K_1^\text{GL}(\Omega(A)((t)))
\end{array}
$$

**Proposition 4.** If $k$ is a field, then there is a canonical isomorphism $K_1^\text{GL}(R((t))) \cong k^*$. This is established by applying the Mayer-Vietoris sequence [5] to the Cartesian square

$$
\begin{array}{ccc}
(\Omega k)^+ & \to & (Ek)^+ \\
\downarrow & & \downarrow (x \mapsto 1)^+ \\
Z & \to & k^+
\end{array}
$$

2. **Tame symbols.** Let $K$ be a field with discrete valuation $v:K^* \to \mathbb{Z}$, valuation ring $A$ and residue class field $L$. Then the tame symbol [2] is a Steinberg symbol [3], [5]:

$$U(K) \times U(K) \xrightarrow{\cdot,} U(L) = L^*$$

defined as follows. Let $v(\pi) = 1$, and let $u, u_1 \in U(K)$. Write $u = a\pi^i$, $u_1 = b\pi^j$ where $a, b \in U(A)$. Then

$$(u, u_1)_v = (-1)^{ij} \overline{a^i/b^j}$$

where "bar" is the residue class in $L$.

If $A$ is a commutative ring, then there is a Steinberg symbol

$$U(A) \times U(A) \to K_2(A)
\begin{array}{ccc}
(u, u_1) & \to & u * u_1
\end{array}$$

defined by $u * u_1 = [h_{12}(u), h_{13}(u_1)]$. [5] and the symbol is natural with respect to ring homomorphisms. By the theorem of Matsumoto...
there is a unique homomorphism $S_v: K^*_2(K) \to L^*$ such that the following diagram commutes

$$
\begin{array}{ccc}
U(K) \times U(K) & \overset{\cdot \cdot}{\longrightarrow} & K^*_2(K) \\
\downarrow \ (,)_v & & \downarrow \omega \\
L^* & \overset{\sim}{\longleftarrow} & \mathbf{K}_0(\Omega k^+). \\
\end{array}
$$

In the case of $k((t)) = K$, $A = k[[t]]$, $L = k$ we have

**Proposition 5.** If $k$ is a field, the following diagram commutes up to sign

$$
\begin{array}{ccc}
U(k((t))) \times U(k((t))) & \overset{\cdot \cdot}{\longrightarrow} & K^*_2(k((t))) \\
\downarrow \ (,)_v & & \downarrow \omega \\
k^* & \overset{\sim}{\longleftarrow} & \mathbf{K}_0(\Omega k^+). \\
\end{array}
$$

**Corollary.** $(,)_v$ factors through $\kappa^*_2(k((t)))$.

Suppose now that $p$ is an irreducible polynomial in $k[t]$, determining the valuation ring $A_p$ over $k$ in $k(t)$. Complete $A_p$ in the $p$-adic topology to get the valuation ring $\tilde{A}$, with residue class field $L = k[[t]]/(p)$ and field of quotients $k(t)$. By the structure theorem for complete local rings [6], $\tilde{A} \cong L[[p]]$ and we have maps $k(t) \to k(t) \cong L((p))$ yielding a commutative diagram

$$
\begin{array}{ccc}
U(k(t)) \times U(k(t)) & \overset{(,)_p}{\longrightarrow} & U(L) \\
\downarrow & & \uparrow (,)_p \\
U(L((p))) \times U(L((p))) & & \\
\end{array}
$$

This diagram, together with the corollary to Proposition 5 applied to $L((p))$ implies

**Proposition 6.** The tame symbol $(,)_p$ on $k(t)$ determined by the irreducible polynomial $p \in k[t]$ factors through $\kappa^*_2(k((t)))$. That is, there is a commutative diagram

$$
\begin{array}{ccc}
U(k(t)) \times U(k(t)) & \longrightarrow & K^*_2(k(t)) \\
\downarrow \ (,)_p & & \downarrow \psi \\
L^* & \overset{\sim}{\longrightarrow} & \kappa^*_2(k(t)) \\
\end{array}
$$

where $L = k[[t]]/(p)$.

As a consequence of Proposition 6 and results of [2] we have
THEOREM 2. If $k$ is a field, there is a split exact sequence

$$0 \to \kappa_2^\text{al}(k) \to \kappa_2^\text{al}(k(t)) \to \bigoplus U(k[t]/(p)) \to 0$$

where the sum is taken over all monic irreducible polynomials $p$ in $k[t]$.

I suspect that an analogue of Proposition 6 is valid for any global field. However, one of the main tools in this work, the results of [1], is not available for discrete valuation rings in the unequal characteristic case.

REFERENCES


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