TRUNCATION ERROR BOUNDS FOR $\pi$-FRACTIONS

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1. Preliminaries. The purpose of this note is to state extensions of the results given in [2] for $g$-fractions. These extensions will be useful for a unification of the theory of inclusion regions for continued fractions associated with certain Hilbert transforms

$$f(z) = \int_{-\infty}^{+\infty} \frac{d\sigma(t)}{z - t}.$$ 

For related results see [1], [3], and [4].

For $-\infty < a < b < +\infty$ let $W(a, b)$ denote the class of nonrational real analytic functions $f(z)$ which are holomorphic for $z \in \text{comp}[a, b]$ and which satisfy $\text{Re}[(z-a)(z-b)]^{1/2}f(z)>0$ in this domain. The principal branch of the square root is assumed.

**Theorem 1.** The following alternative characterizations of the class $W(a, b)$ are valid:

(a) $f \in W(a, b)$ if and only if there is a bounded nondecreasing function $\sigma$, with infinitely many points of increase, such that

$$f(z) = \int_{a}^{b} \frac{d\sigma(t)}{z - t}, \quad z \in \text{comp}[a, b];$$

(b) $f \in W(a, b)$ if and only if $f$ has a (unique) $\pi$-fraction expansion

$$f(z) = \begin{cases} \pi_0 + \frac{b - a}{z - b} + \frac{\pi_1(z - a)}{z - b} + \frac{b - a}{z - b} \\ + \frac{\pi_2(z - a)}{z - b} + \cdots, \quad z \in \text{comp}[a, b], \end{cases}$$

with $\pi_n > 0$, $n \geq 0$.

2. Inclusion regions. The first inclusion theorem is a consequence of Theorem 1(a).

**Theorem 2.** If $f \in W(a, b)$ and $z$ is nonreal then $f(z)$ is contained in the open convex sector $K_{-\lambda}(z)$ bounded by the rays

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1091
\[ K_{-1}(z) : w = \pi/(z - b), \quad k_{-1}(z) : w = \pi/(z - a), \quad 0 \leq \pi \leq +\infty. \]

\( K_{-1}(z) \) is precisely the set of all first approximants

\[ w^b_1(z) = \frac{\pi'}{z - b} + \frac{b - a}{1} + \frac{\pi^*(z - a)}{z - b} \quad (\pi' > 0, \pi^* > 0), \]

or

\[ w^a_1(z) = \frac{\pi}{z - b} + \frac{b - a}{1} + \frac{\pi(z - a)}{z - b} + \frac{b - a}{1} \quad (\hat{\pi} > 0, \hat{\pi} > 0), \]

of \( \pi \)-fractions (1).

This result can now be extended to provide inclusion regions \( K_n(z) \) which contain \( f(z) \), and which are best possible if the first \( n + 1 \) coefficients \( \pi_0, \pi_1, \ldots, \pi_n \) are known. For \( z \) nonreal the linear fractional transformations

\[ t_n(w) = \frac{\pi_n}{z - b} + \frac{b - a}{1 + \pi(z - a)w} \quad (n \geq 0) \]

are nonsingular with determinants \( \pi_n(b - a)(s - a) \). Let the composed transformations

\[ T_n(w) \equiv t_0 \circ t_1 \circ \cdots \circ t_n(w) \quad (T_{-1}(w) \equiv w), \]

and define

\[ K_n(z) = T_n(K_{-1}(z)) \quad (n \geq -1). \]

The transformations \( T_n \) are also nonsingular linear fractional transformations. From Theorem 2, \( K_n(z) \) is the intersection of two circular disks. Moreover the geometry of the sets \( K_n(z) \) may be described completely in terms of the approximants

\[ w^b_0(z), w^a_0(z), w^b_1(z), w^a_1(z), w^b_2(z), \ldots \]

of the \( \pi \)-fraction (1).

**Theorem 3.** Let \( z \) be nonreal, and let \( f \in W(a, b) \) have the \( \pi \)-fraction expansion (1) with approximants (2) and associated sets \( K_n(z(n \geq -1, w^b_{-1}(z) = \infty, w^a_{-1}(z) = 0) \). Then the following statements are true for \( n \geq 0 \).

(a) \( f(z) \in K_n(z) \).

(b) \( K_n(z) \) is precisely the set of all \( (n + 2) \)th approximants \( w^b_{n+2}(z) \), \( w^a_{n+2}(z) \) of \( \pi \)-fractions (1) with \( \pi_0, \pi_1, \ldots, \pi_n \) fixed.

(c) \( K_n(z) \) is open, bounded, and convex with interior angles

\[ \theta \equiv \left| \arg[(z - b)/(z - a)] \right|. \]
(d) \( K_n(z) \subset K_{n-1}(z) \).
(e) \( K_{n-1}(z) - K_n(z) \) consists of two components \( L_a^b(z) \) and \( L_b^a(z) \). \( L_a^b(z) \) (\( L_b^a(z) \)) is a circular triangle with vertices

\[ w_n^a(z), \, w_n^b(z), \, w_n^a(z) \quad (w_{n-1}^b(z), \, w_n^a(z), \, w_n^b(z)), \]

and respective interior angles \( \theta, \alpha = |\arg(z-a)|, \beta = |\arg(b-z)| \).

The following limiting case of Theorem 3 is a consequence of (e).

**Corollary.** If \( z = x > b \) then

\[ w_0^a(x) < w_1^a(x) < w_2^a(x) < \cdots < w_k^a(x) < w_1^b(x) < w_0^b(x), \]

and if \( z = x < a \) then

\[ w_0^a(x) < w_1^b(x) < w_2^a(x) < \cdots < w_k^b(x) < w_1^a(x) < w_0^b(x). \]

3. **A priori bounds.** The theory of continued fractions, the special form of (1), and the inequality between the arithmetic and geometric means now provide bounds for \( w_n^a(z) - w_n^b(z) \), and hence also for the diameter of \( K_n(z) \). Furthermore special examples show that the rate of convergence implied by these bounds is best possible over the class \( W(a, b) \).

**Lemma.** The function

\[ \rho(z) = \frac{(z - a)^{1/2} - (z - b)^{1/2}}{(z - a)^{1/2} + (z - b)^{1/2}} = \frac{1 - \left( \frac{z - b}{z - a} \right)^{1/2}}{1 + \left( \frac{z - b}{z - a} \right)^{1/2}} \]

\[ = \frac{(z - a) - 2((z - a)(z - b))^{1/2} + (z - b)}{b - a} \]

maps the domain \( \text{comp}[a, b] \) conformally onto the open unit disk: \( |\rho(z)| < 1 \) for \( z \in \text{comp}[a, b] \).

**Theorem 4.** For \( z \in \text{comp}[a, b] \) the diameter of \( K_n(z) \) satisfies the inequality

\[ \text{diam} K_n(z) \leq \frac{\pi_0 |\rho(z)|^n}{|z - a| |z - b| \kappa(\theta)} \quad (n \geq 0) \]
with

\[ \kappa(\theta) \equiv \cos \frac{\theta}{2} \begin{cases} 
1, & 0 \leq \theta \leq \frac{\pi}{2}, \\
\sin \theta, & \frac{\pi}{2} \leq \theta < \pi,
\end{cases} \]

\[ \theta \equiv \left| \frac{z - b}{z - a} \right|. \]

Moreover

\[ \sup_{r \in W(a, b)} \limsup_{n \to \infty} \left[ \text{diam } K_n(z) \right]^{1/n} = \left| \rho(z) \right|. \]

REFERENCES


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