ONE-PARAMETER SEMIGROUPS OF ISOMETRIES

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Communicated by I. M. Singer, March 2, 1970

Let \( t \to V_t \) for \( t \geq 0 \) be a strongly continuous one-parameter semigroup of isometries on a Hilbert space \( H \). The easiest example of such a semigroup which is not unitary is given by considering the Hilbert space \( \tilde{H} = L^2(0, \infty) \) consisting of those Lebesgue square-integrable functions on \( (-\infty, \infty) \) which are supported on \( (0, \infty) \). On \( \tilde{H} \), we consider the (nonunitary) isometries

\[
(T_t f)(x) = f(x - t).
\]

Recently, the C*-algebra \( \mathcal{A}(T_t : t \geq 0) \) generated by the semigroup \( t \to T_t \) has been studied in detail [2], [3], [4].

In this note, we show that for any strongly continuous one-parameter semigroup of isometries \( t \to V_t \) with \( V_{t_0} \) not unitary for some \( t_0 \), \( \mathcal{A}(V_t : t \geq 0) \) is *-isomorphic with \( \mathcal{A}(T_t : t \geq 0) \). The proof is modelled after the corresponding result for C*-algebras generated by a single isometry [1].

The main fact that we use is a generalization due to Cooper [6, p. 142] of the Wold decomposition of a single isometry [5, p. 109]. This generalization states that for \( t \to V_t \) with \( V_{t_0} \) not unitary for some \( t_0 \), there is a Hilbert space \( K \) with a strongly continuous one-parameter unitary semigroup \( t \to U_t \) on \( K \), there is a cardinal \( \alpha \), and there is an isometry \( U \) from \( H \) onto \( K \oplus \tilde{H} \oplus \cdots \oplus \tilde{H} \oplus \cdots \) where \( \tilde{H} \) occurs with multiplicity \( \alpha \), such that

\[
UV_tU^* = U_t \oplus T_t \oplus \cdots \oplus T_t \oplus \cdots
\]

The multiplicity \( \alpha \) is a unitary invariant which can be read off from the infinitesimal generator of \( t \to V_t \) [6, p. 142].

In case \( K = \{0\} \), we say that \( t \to V_t \) is purely nonunitary [6, p. 136]. For such semigroups, the multiplicity \( \alpha \) is the only unitary invariant. A very general way of generating such semigroups is to consider for any measure \( d\mu \) which is positive, of bounded variation, and singular with respect to Lebesgue measure on the unit circle \( T \), the singular inner functions [5, p. 66] \( \phi^*(e^{it}) \) which are the boundary values of

AMS 1969 subject classifications. Primary 4665, 4750.

Keywords and phrases. C*-algebras, semigroups of operators.
It is then easy to check that for $f$ in the usual Hardy space $H^p(T)$ \cite[p. 39]{5},
\[(M_t^\mu)(e^{\theta}) = \phi_t(e^{\theta})f(e^{\theta})\]
defines a strongly continuous one-parameter semigroup of isometries for $t \geq 0$. The second result of this note shows that $t \mapsto M_t^\mu$ is purely nonunitary and characterizes the multiplicity $\alpha(\mu)$ of $t \mapsto M_t^\mu$ directly in terms of the measure $\mu$.

**Theorem A.** Let $t \mapsto V_t$, $t \geq 0$, be a strongly continuous one-parameter semigroup of isometries with $V_{t_0}$ nonunitary for some $t_0$. Then the $C^*$-algebra $\mathcal{A}(V_t : t \geq 0)$ generated by the $V_t$ is *-isomorphic with $\mathcal{A}(T_t : t \geq 0)$.

**Proof.** Applying the decomposition of Cooper to $t \mapsto V_t$, we see that the problem is reduced to studying
\[
\alpha = \mathcal{A}(U_t \oplus T_t \oplus \cdots \oplus T_t \oplus \cdots : t \geq 0),
\]
where $T_t$ occurs with multiplicity $\alpha \geq 1$. Now $\alpha$ is just the norm-closure of direct sums of the form
\[
\sum_{j=1}^n a_{t_j, s_j} U_{t_j} U_{s_j}^* \oplus \sum_{j=1}^n a_{t_j, s_j} T_{t_j} T_{s_j}^* \oplus \cdots .
\]
The mapping $\Phi$ which sends such a direct sum to
\[
\sum_{j=1}^n a_{t_j, s_j} T_{t_j} T_{s_j}^*
\]
clearly extends to a *-homomorphism from $\alpha$ onto $\mathcal{A}(T_t : t \geq 0)$. To check that $\Phi$ is actually a *-isomorphism, it suffices to show that
\[
\left\| \sum_{j=1}^n a_{t_j, s_j} U_{t_j} U_{s_j}^* \right\| \leq \left\| \sum_{j=1}^n a_{t_j, s_j} T_{t_j} T_{s_j}^* \right\| .
\]
The structure of the algebra $\mathcal{A}(T_t : t \geq 0)$ has been described in \cite{2}, \cite{3}. We use the fact that $\mathcal{A}(T_t : t \geq 0)$ contains a proper closed two-sided ideal $\mathbb{C}$ (the commutator ideal) and for $R$ the real line,
\[
\inf_{C \in \mathbb{C}} \left\| \sum_{j=1}^n a_{t_j, s_j} T_{t_j} T_{s_j}^* + C \right\| = \sup_{x \in R} \left\| \sum_{j=1}^n a_{t_j, s_j} \exp[i(t_j - s_j)x] \right\| .
\]
It follows that it will be enough to show that
Now noting that \( t \mapsto U_t \) is a strongly continuous semigroup for \( t \geq 0 \), we see that \( U_t \) commutes with \( U_t^* \) and \( t \mapsto U_t \) can be extended to a unitary representation of \( R \) by defining \( U_t = U_t^* \) for \( t \geq 0 \). The desired inequality is obtained by observing that for some self-adjoint (not necessarily bounded) \( A \) on \( H \),
\[
\langle U_t f, g \rangle = \int_{x \in \sigma(A)} e^{itx} \bar{d}(E(x)f, g), \quad t \in (-\infty, \infty),
\]
where \( f \) and \( g \) are in \( H \), \( A \) is the infinitesimal generator for \( t \mapsto U_t \), \( E(x) \) is the spectral family for \( A \), and \( \sigma(A) \) is the spectrum of \( A \) \((\sigma(A) \subset R) [6, p. 134]\). Hence, using the fact that for \( \|f\| = 1 \)
\[
\int_{x \in \sigma(A)} d\langle E(x)f, f \rangle = 1,
\]
we see that for \( \|f\| = 1 \)
\[
\left\| \sum_{j=1}^{n} a_{t_j, s_j} U_{t_j} U_{s_j}^* \right\|^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} d_{t_j, s_j} a_{t_k, s_k} \langle U_{s_j-t_j+t_k-s_k} f, f \rangle
\]
\[
= \int_{x \in \sigma(A)} \left| \sum_{j=1}^{n} a_{t_j, s_j} \exp[i(t_j - s_j)x] \right|^2 d\langle E(x)f, f \rangle
\]
and the desired inequality follows.

**Theorem B.** The strongly continuous one-parameter semigroup of isometries \( t \mapsto M_t \) described above is purely nonunitary and the multiplicity \( \alpha(\mu) \) is determined as follows: \( \alpha(\mu) = n \) if the support of \( \mu \) consists of exactly \( n \) points, \( \alpha(\mu) = \infty \) otherwise.

**Proof.** Let us first show that if \( w \) is any nonconstant inner function \([5, p. 62]\), then for \( f \) in \( H^2(T) \), the isometry \( (M_w f)(z) = w(z) f(z) \) is purely nonunitary. Otherwise, for some \( f \) in \( H^2(T) \) with \( \|f\| = 1 \), \( \|M_w^* f\| = 1 \) for \( n = 1, 2, \cdots \), or equivalently, for \( g_n \) in \( H^2(T) \)
\[(*) \quad f = w^n g_n, \quad n = 1, 2, \cdots .\]
Thus, if \( w(z_0) = 0 \) for \( |z_0| < 1 \) then \( f \) has a zero of infinite order at \( z_0 \), which is impossible. Thus, \( w \) is purely singular and nonconstant so
\[
w(z) = \exp \left\{ - \int \frac{e^{i\alpha} + z}{e^{i\alpha} - z} \, d\nu(\alpha) \right\}
\]
where \( \nu \) is a uniquely determined finite positive singular measure on
Equating the singular parts of the functions in (*), we see that for

\[ f_{\text{sing}}(z) = \exp \left\{ - \int \frac{e^{i\alpha} + z}{e^{i\alpha} - z} \, d\sigma(\alpha) \right\} , \]

\[ (g_n)_{\text{sing}}(z) = \exp \left\{ - \int \frac{e^{i\alpha} + z}{e^{i\alpha} - z} \, d\tau_n(\alpha) \right\} , \]

where \( \sigma \) and \( \tau_n \) are finite positive singular measures on \( T \), we have \( \sigma = \nu + \tau_n \) so \( \sigma(T) \geq \nu(T) \) for \( n = 1, 2, \cdots \). Since \( \sigma(T) < \infty \) and \( \nu(T) > 0 \), we have a contradiction.

We remark further that the defect of \( M_w \) (the dimension of kernel(\( M^* \))) is finite if and only if \( w \) is a finite Blaschke product, and that in this case the defect equals the number of terms in the Blaschke product. We now prove this assertion. Certainly, if \( w = \prod_{k=1}^{N} w_k \), where the \( w_k \) are nonconstant inner functions, then \( M_w = \prod_{k=1}^{N} M_{w_k} \).

Each of the \( M_{w_k} \) are purely nonunitary isometries and so have defect at least one. Thus, since

\[ \text{defect}(M_w) = \sum_{k=1}^{N} \text{defect}(M_{w_k}) \]

(this follows from elementary index-type argument), we have defect \( (M_w) \geq N \). It follows easily that if \( w \) has a nonconstant singular part or a Blaschke part with infinitely many zeros, \( \text{defect}(M_w) = \infty \). If

\[ w(z) = \lambda \prod_{k=1}^{N} \left( \frac{z - a_k}{1 - \bar{a}_k z} \right) \]

where \( \lambda \) is a constant, \( |\lambda| = 1 \) and \( |a_k| < 1 \), then

\[ \text{defect}(M_w) = \sum_{k=1}^{N} \text{defect}(M_{(\langle e^{-a_k} \rangle)/(1-\bar{z}_a)}) \]

and each \( M_{(\langle e^{-a_k} \rangle)/(1-\bar{z}_a)}) \) has defect one.

Now by the result of Foiaș and Nagy [6, p. 142], \( \alpha(\mu) \) is just the defect of the isometry obtained by taking the Cayley transform of the infinitesimal generator of the semigroup \( t \mapsto M_t^\phi \). The infinitesimal generator of the semigroup is \( M_\psi \), where \( \psi \) is the function

\[ \psi(z) = -\int \frac{e^{i\alpha} + z}{e^{i\alpha} - z} \, d\mu(\alpha) , \quad |z| < 1. \]

The Cayley transform of \( M_\psi \) is \( M_w \), where \( w = (1 + \psi)/(1 - \psi) \). Since Re\( \psi(z) < 0 \), we see that \( |w(z)| < 1 \) for \( |z| < 1 \). Since \( \mu \) is singular,
so \( w \) is an inner function.

By the foregoing, it suffices to show that \( w \) is a finite Blaschke product of \( n \) terms if and only if \( \text{support}(\mu) \) is a finite set with \( n \) points. But if \( \text{support}(\mu) \) is finite (with \( n \) points) then \( w(z) \) is a rational function. The only rational inner functions are finite Blaschke products [5, p. 76] so \( w \) has the form

\[
w(z) = \lambda \prod_{k=1}^{m} \left( \frac{z - a_k}{1 - \bar{a}_k z} \right)
\]

where \( |\lambda| = 1 \) and \( |a_k| < 1 \). But the zeros of \( w(z) \) in the plane are those of

\[
1 + \psi(z) = 1 - \sum_{k=1}^{n} \frac{\exp[i\alpha_k] + z}{\exp[i\alpha_k] - z} t_k, \quad t_k > 0,
\]

and multiplying by \( \prod_{k=1}^{n} (\exp[i\alpha_k] - z) \) we see that \( 1 + \psi(z) \) has exactly \( n \) zeros so that \( n = m \). Conversely, suppose

\[
w(z) = \lambda \prod_{k=1}^{n} \left( \frac{z - a_k}{1 - \bar{a}_k z} \right), \quad |\lambda| = 1.
\]

Then

\[
- \int \frac{e^{i\alpha} + z}{e^{i\alpha} - z} d\mu(\alpha) = \psi(z) = \frac{w(z) - 1}{1 + w(z)}
\]

is a rational function and so has at most finitely many points of \( T \) in its natural boundary. But then \( \text{support}(\mu) \) contains only those points [5, p. 68], and so is finite. This completes the proof.

REFERENCES


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