A CHARACTERIZATION THEOREM 
FOR CELLULAR MAPS

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Introduction. The main result of this paper is that a mapping \( f \) of the \( n \)-sphere \( \partial B^{n+1}, n \neq 4 \), onto itself is cellular if and only if \( f \) has a continuous extension which maps the interior of the \( n+1 \) ball \( B^{n+1} \) homeomorphically onto itself. Since a map of a 2-sphere onto itself is cellular if and only if it is monotone, this theorem extends a result of Floyd and Fort [6], who prove the corresponding theorem for monotone maps on a 2-sphere.

Preliminaries. A compact mapping \( f : M^n \to X \) is cellular if for each \( x \in X \), there is a sequence \( C_1, C_2, \ldots \) of topological \( n \)-cells such that \( f^{-1}(x) = \bigcap_{i=1}^\infty C_i \) and \( C_{i+1} \subset \text{Int}C_i \). If \( X \) is a topological space, \( H(X) \) is the group of all homeomorphisms of \( X \) onto itself. Edwards and Kirby showed that for any compact manifold \( M \), \( H(M) \) is locally contractible and therefore uniformly locally arcwise connected. It was shown [7] that any mapping of a manifold onto itself which can be uniformly approximated by homeomorphisms is cellular. (See also [4] by Armentrout \( n = 3 \) [1] and Siebenmann \( n \geq 5 \) [10] have proven that any cellular mapping of a manifold onto itself can be uniformly approximated by homeomorphisms.

Lemma 1. Suppose \( f : \partial B^n \to \partial B^n \) can be approximated by homeomorphisms. Then \( f \) can be extended to a map which is a homeomorphism on the interior of \( B^n \).

Proof. Since \( f \) can be uniformly approximated by homeomorphisms and \( H(\partial B^n) \) is uniformly arcwise connected, there is an arc \( \Phi \) such that \( \Phi_t = f \) and \( \Phi_0 \in H(\partial B^n) \), for \( 0 \leq t < 1 \). Each point of \( B^n \) can be represented in the form \( tx \), where \( x \in \partial B^n \) and \( 0 = t = 1 \). We define \( F : B^n \to B^n \) by \( F(tx) = t\Phi_t(x) \), for all \( x \in \partial B^n \). We note that \( F \) is continuous, extends \( f \) and is a homeomorphism when restricted to the interior of \( B^n \).

Therefore, if \( n \neq 4 \) and \( f : \partial B^{n+1} \to \partial B^{n+1} \) is cellular \( f \) can be extended to a map which is a homeomorphism on the interior of \( B^{n+1} \).

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A map has property \( UV^\infty \) if for each \( x \) and each open set \( U \) containing \( f^{-1}(x) \), there is an open set \( V \) containing \( f^{-1}(x) \) such that \( V \subseteq U \) and \( V \) is null-homotopic in \( U \).

**Lemma 2.** Let \( M \) be a manifold and \( F : M \times (0, 1] \to M \times (0, 1] \) be a map such that \( F^{-1}(M \times 1) = M \times 1 \) and \( F/M \times (0, 1) : M \times (0, 1) \to M \times (0, 1) \) is a homeomorphism, then \( F/M \times 1 : M \times 1 \to M \times 1 \) is a \( UV^\infty \) map.

**Proof.** We identify \( M \) with \( M \times 1 \). We make use of the following auxiliary maps: for each \( \partial \), define \( \pi_\partial : M \to M \times (1 - \partial) \) by \( \pi_\partial(x) = (x, 1 - \partial) \) and \( \rho : M \times (0, 1] \to M \) by \( \rho(x, t) = (x, 1) = x \).

Let \( U' \) be open in \( M \) with \( f^{-1}(b) \subseteq U' \). \( U' \times (0, 1] \) is open in \( M \times (0, 1] \). Therefore, there is a \( U \) such that:

1. \( U \) is open in \( M \times (0, 1] \).
2. \( U \subseteq U' \times (0, 1] \).
3. \( f(U) \) is open in \( M \times (0, 1] \).
4. \( f^{-1}(b) \subseteq U \).

Now choose \( t_0 < 1 \) and an open cylinder, \( C \), about \( b \times [t_0, 1] \) such that \( C \subseteq \{f(U)\} \). We note that:

\( f^{-1}(C) \) is open in \( M \times (0, 1] \), \( f^{-1}(C) \subseteq U \), \( f^{-1}(b \times [t_0, 1]) \subseteq f^{-1}(C) \).

Let \( \eta = \delta(b, C) ; \eta > 0 \). Let \( \delta \) be chosen so that

1. \( N_\delta(f^{-1}(b)) \subseteq f^{-1}(C) \).
2. \( d(x, y) < 2\delta \Rightarrow d(f(x), f(y)) < \eta \).

Let \( V = N_\delta(f^{-1}(b)) \cap M \). We note that if \( x \) is an element of \( \pi_\partial(V) \), then \( f(x) \) is an element of \( N_\delta(b) \cap M \times (0, 1) \subseteq C \).

Since \( C \) is a cell we can define a homotopy \( G : C \times I \to C \) so that

1. \( x \in C \Rightarrow G(x, t) \in C \cap (M \times (0, 1)) \).
2. \( G(x, 0) = x \).
3. \( \exists s \in M \times (0, 1) \) such that \( G(x, 1) = s \), for all \( x \in C \).

We now can define the desired homotopy \( H : V \times I \to U' \), by \( H(x, t) = pf^{-1}(G(f\pi_\partial(x), t)) \). Thus, \( H(x, 0) = pf^{-1}[G(f\pi_\partial(x), 0)] = pf^{-1}(f\pi_\partial(x)) = x \).

\[ H(x, 1) = pf^{-1}[G(f\pi_\partial(x), 1)] = pf^{-1}(x) = \text{constant}. \]

The continuity of \( f \) follows from that of \( G \), so all that remains to be shown is that \( H(x, t) \in U' \), for all \( x \in V \), \( \forall t \in I \).

\[ x \in V \Rightarrow \pi_\partial(x) \in \pi_\partial(V) \Rightarrow f(\pi_\partial(x)) \in C \cap M \times (0, 1) \]
\[ \Rightarrow G(f\pi_\partial(x), t) \in C \cap B^\partial \Rightarrow \]

that \( f^{-1} \) is defined and \( f^{-1}[G(f\pi_\partial(x), t)] \in f^{-1}(C) \subseteq U \subseteq U' \times (\frac{1}{2}, 1] \).

Thus \( P(f^{-1}[G(f\pi_\partial(x), t)]) = H(x, t) \in U' \).
Let $M \subset X$. $M$ is collared if there is a homeomorphism $h : M \times (0, 1] \to \text{nbd of } M$ such that $h(m, 1) = m$, for all $m \in M$. M. Brown proved that the boundary of any manifold with boundary is collared [3]. Therefore, we have the following corollary.

**Corollary.** Let $M$ be a manifold with boundary and let $f : M \to M$ be such that $f$ restricted to the interior of $M$ is a homeomorphism. Then $f/\partial M$ is a UV$^\infty$-map.

Using McMillan's criteria for cellularity, [9] it can easily be shown that if $f : M^n \to M^n$ is a UV$^\infty$-map and if $M^n = S^3$ or $n \geq 5$, then $f$ is a cellular map. (Cf., Armentrout and Price [2] or Lacher [8].) We therefore have the following theorem:

**Theorem.** A mapping $f$ of the $n$-sphere $\partial B^{n+1}$, $n \neq 4$, onto itself is cellular iff $f$ has a continuous extension which maps the interior of $B^n$ homeomorphically onto itself.

**Corollary.** Let $M$ be an $m$-manifold, $n \geq 5$, with boundary. Let $f$ be a map of $M$ onto $M$ such that $f/\text{Int } M : \text{Int } M \to \text{Int } M$ is cellular and $f/\partial M : \partial M \to \partial M$. Then $f/\partial M$ is a UV$^\infty$ map. In particular, if $n \geq 6$, $f/M$ is a cellular map.

**Proof.** Define $g : \text{Int } M \to (0, \infty)$ by $g(m) = d(m, \partial M)$. Since $f/\text{Int } M$ is a cellular map, by Siebenmann’s theorem there is a homeomorphism $h$ such that for all $x \in \text{Int } M$, $d(f(x), h(x)) < g(f(x))$. We define $F : M \to M$ by

$$F(x) = h(x), \quad x \in \text{Int } M,$$

$$= f(x), \quad x \in \partial M.$$

$F$ is continuous, for suppose there is a sequence, $x_n$, of points in $\text{Int } M$ which converge to $x \in \partial M$. Let $\epsilon > 0$ be given. By the continuity of $f$, $\exists N \ni n > N \Rightarrow d(f_n(x), f(x)) < \epsilon/2$. Then for such $n$,

$$d(F(x_n), F(x)) = d(h(x_n), f(x)) \leq d(h(x_n), f(x_n)) + d(f(x_n), f(x)) < \epsilon.$$

Thus, by Lemma 2, $F/\partial M = f/\partial M$ is a UV$^\infty$ map.

Armentrout’s approximation theorem [1] and results of E. E. Floyd [5] make it possible to prove the corresponding result for three manifolds: For such $M$, if $f : M \to M$ is a proper map such that $f/\text{Int } M$ is cellular, then $f/\partial M$ is cellular.

**References**


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