THE ORDER OF THE IMAGE OF THE 
J-HOMOMORPHISM

BY MARK MAHOWALD

Communicated by Raoul Bott, June 4, 1970

ABSTRACT. This note announces a proof of the order of the 
image of the J-homomorphism and gives several other results in 
homotopy theory which are consequences of the proof.

The set $\Omega^nS^n$ can be identified with the set of all base point pre­ 
serving maps of $S^n$ into itself. $SO(n)$, acting on $S^n$ as $R^n$ with a point 
at infinity, is also a set of base point preserving maps of $S^n$ onto itself. This defines $SO(n) \subset \Omega^nS^n$. The induced map in homotopy is 
called the J-homomorphism. If we allow $n$ to go to infinity we have 
the stable J-homomorphism. By Bott’s results $[3]$ $\pi_j(SO) = Z, j \equiv -1 \mod 4, and =Z_2j=0, 1 \mod 8, j>0, and zero otherwise.

Adams $[1]$ showed that the $Z_2$ summand maps monomorphically 
and Milnor and Kervaire $[6]$ showed that the $Z$ group in dimension 
$4j-1$ maps nontrivially and its image generates a subgroup of at 
est least a certain order $\lambda_j$. Adams $[1]$ showed that the order was either 
$\lambda_j$ or $2\lambda_j$ and if $j \equiv 1 \mod 2$ it was $\lambda_j$. Thus only the two primary part is 
in question and there only for $j \equiv 0 \mod 2$. Let $\lambda_j$ be the two primary 
part of $\lambda_j$. If $4j \equiv 2\rho(j) \mod 2\rho(j)+1$ (which defines $\rho(j)$) then $\lambda_j = 2\rho(j)+1$.

We prove:

**THEOREM 1.** The 2-primary order of the image of J in stem $4j-1$ is $\lambda_j$.

The proof has several corollary results which have some interest. 
The first result is rather technical but still has some interest. The 
naming of elements in $H^{**}(A)$ is that given in $[5]$.

**THEOREM 2.** The elements $P_{i}c_0, P_{i}h_1c_0, i \geq 1, P_{i}h_2, i \geq 1$, in $H^{**}(A)$ 
represent the image of J in dimension $j \equiv 0, 1, 3 \mod 8$. In dimension 
$8j-1$ the “tower” which ends at the “Adams edge” represents the image 
of J in that dimension.

These elements were known to have the desired $e$-invariant 
property $[1]$ and were believed to be in J. Their Whitehead product 
behavior has been investigated ([2] and [4], for example).

Let $M = Z_2 + Z_2$ (be the module over $A$ with one generator; $\mu$ in

**AMS 1970 subject classifications.** Primary 55E10, 55E50, 55H15.

**Key words and phrases.** Stable homotopy groups of spheres, J-homomorphism, 
cohomology of the Steenrod algebra.

1310
THE ORDER OF THE IMAGE OF THE J-HOMOMORPHISM

Let \( P(x_1, \cdots) \) be a polynomial algebra on generators \( x_i \) with bidegree \((2, 2i+2+1)\). Consider the differential \( d(x_i) \rightarrow x_{i-1}^2 x_1 \) in \( P \). Let \( H(d) \) be the homology under \( d \) and \( B(d) = \text{im} \, d \).

For \( \alpha \in P \) let the bidegree of \( \alpha \) be \((\alpha_s, \alpha_t)\). We will be only interested in the values of \( \alpha_s \) modulo 4 and \( \alpha_t \) modulo 12 so take \((\alpha_s, \alpha_t)\) so that \( \alpha_s \equiv \alpha'_s \pmod{4} \), \( \alpha_t \equiv \alpha'_t \pmod{4} \) but \( 5\alpha_s \leq \alpha_t - \alpha_s \).

**Theorem 3.** If \( 5s \geq t - s + \epsilon \) where \( \epsilon \) depends on the congruence class of \( s \) mod 4 and \( \epsilon \leq 6 \), then

\[
\text{Ext}^s_{A}(M, \mathbb{Z}_2) = \sum_{\alpha \in H(d)} \text{Ext}^{s-\alpha_s, t-\alpha_t}_{A}(M \otimes A/ A_1, \mathbb{Z}_2)
\]

\[
\bigoplus \sum_{\alpha \in B(d)} \text{Ext}^{s-\alpha_s, t-\alpha_t}_{A}(M \otimes A/ A(Sq^1, Sq^2), \mathbb{Z}_2).
\]

**Corollary 4.** If \( Q \) is an \( A \) module which is free over \( A_1 \), the subalgebra generated by \( Sq^1 \) and \( Sq^2 \), then \( \text{Ext}^s_A(Q, \mathbb{Z}_2) = 0 \) for \( 5s \geq t - s + \epsilon \).

**Theorem 5.** Let \( X \) be a space in the stable category so that \( \Sigma X = RP^2 \). If \( E_r(X) \) is the Adams spectral sequence converging to \( \pi^A_*(X) \), then \( E^s_r(X) = 0 \) for \( 5s \geq t - s + \epsilon \) unless

\[
s = 4k, \quad t - s = 8k, \quad 8k + 1, \quad 8k + 2,
\]

\[
= 4k + 1, \quad t - s = 8k + 1, \quad 8k + 2, \quad 8k + 3,
\]

\[
= 4k + 2, \quad t - s = 8k + 2, \quad 8k + 3, \quad 8k + 7,
\]

\[
= 4k + 3, \quad t - s = 8k + 4, \quad 8k + 8, \quad 8k + 9,
\]

in which cases the groups are \( \mathbb{Z}_2 \).

These elements represent the generators of the image of \( J \) and \( \mu_i \) \cite{1} on the bottom cell and the elements of order two in the \( \text{im} \, J \) and \( \mu_i \) coextended on the top cell.

**Theorem 6.** There is a space \( \text{Im} \, J \) and a map \( f: S^8 \rightarrow \text{Im} \, J \) so that \( f_* \) maps the image of \( J \) and the \( \mu \)'s monomorphically onto the homotopy of \( \text{Im} \, J \).

In \cite{1} a map \( f: \Sigma^8 X \rightarrow X \) which represents an extension of a coextension of \( 8 \sigma \) is studied. There it is proved that all iterations of \( f \) are essential.

**Theorem 7.** If \( \alpha: S^8 \rightarrow X \) is a stable map then

\[
\xymatrix{ S^{k+bj} \ar[r]^{\Sigma^j \alpha} & \Sigma^{bj} X \ar[r]^{f^j} & X }
\]
is inessential for some $j$ unless $\alpha$ is in one of the classes given by Theorem 5.

Some comments on the proof. Let the spectrum $bo$ be the connected $BO$ spectrum. Then we construct a Novikov resolution of $S$ as follows

$$
\cdots \\
S_s \to S_s \wedge bo \\
\cdots \\
S_1 \to S_1 \wedge bo \\
S \to S \wedge bo.
$$

We apply the $E_2$ of the Adams spectral sequence to this tower and get a spectral sequence which converges to $\mathcal{H}^{**}(A)$ except for $s=t$. If we consider the resolution $X \wedge S_s$, where $X$ is defined in Theorem 5, we can make an explicit calculation. Let

$$
E_1^{s,t} = \text{Ext}^t_{A}(\mathcal{H}^{s}(X \wedge S_s \wedge bo), \mathbb{Z}_2).
$$

**Proposition 8.** $E_2^{s,t} = \sum_{a \in \mathbb{F}_2} \text{Ext}^{s-a,t-a}(M \otimes A//A_1, \mathbb{Z}_2)$ for $s>\sigma$ where $P^s$ is the set of $\sigma$-degree polynomials in the polynomial algebra introduced above.

**Proposition 9.** $E_3^{s,t} = E_\infty^{s,t}$ for $s>\sigma$ and thus is given by Theorem 3.

Note that Proposition 9 alone gives an edge of $3\sigma > t-2$. The sharpened version of Theorem 3 follows from Proposition 9 and a closer analysis of the nature of $\text{Ext}_n^t(M, \mathbb{Z}_2)$.

The most direct route from Proposition 8 to the main theorem requires a geometric realization of the $E_2$ term of the above spectral sequence for $S$. Using this resolution and the homotopy functor we get a spectral sequence whose $E_2^{s,t}$ term has an edge of $5\sigma \geq t - \sigma + \epsilon$. The image of $J$ has filtration 1. From this information the order of $\text{im} J$ should follow directly but no direct route has been found. Hence to complete the argument, consider the space $Y$ which is the fiber of the map $S \to K(\mathbb{Z}, 0)$, and consider the resolution of $Y$ given by

$$
\cdots \to Y \wedge S_0 \to Y \wedge S_{-1} \to \cdots.
$$

Calculation of the sort given in the proof of III 7.3 of [4] and applied to elements of filtration zero and one give a proof of Theorems 1 and 2.

**References**


Northwestern University, Evanston, Illinois 60201