any lesser power of technique. When one of the main researchers in a field takes the time and trouble to write an exposition, he makes a valuable contribution; certainly that is the case here.

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Geometric measure theory by Herbert Federer. Springer-Verlag, Berlin, Heidelberg, New York, 1969. 676 pp. $29.50.

Measure theory is perhaps the least honored of the several large mathematical disciplines which have been developed during the twentieth century.

A number of reasons may be given for this humble standing of the subject. In the first place, the French school of mathematicians, with its high prestige level and talent for persuasiveness, had relegated the subject to a relatively minor role and declared it to be a small branch of functional analysis, another discipline of rather low status, except perhaps in its applications to partial differential equations. A second reason is that the subject was largely regarded as a tool for probability theory and this, for a while, involved, for the most part, some of the pleasant but not especially deep or difficult aspects of measure theory. Thirdly, the representation of measure algebras consists primarily of one theorem, that a homogeneous nonatomic normal algebra is homeomorphic to the measure algebra of a product of circles, and this has an interesting but not especially difficult proof.

This is an improper assessment of measure theory. Probability theory has had an unexpectedly proliferous growth, giving more to other fields than it has taken. It is often indistinguishable from measure theory. It may suffice to note its connection with potential theory via brownian motion and more general processes, its elucidation of the behavior of orthonormal systems via martingales, and its generalization of differentiation theory.

A more serious slighting of measure theory by distinguished and influential people lay in their failure to emphasize that the deepest and perhaps most useful aspects of measure theory relate to those measures for which open sets have infinite measure. Geometric measure theory deals with measures of this sort. Recent developments, notable among which are the appearance of Federer's book, are bringing about a change in the position of measure theory described above. The subject deals with measures of lower dimensional sets imbedded in higher dimensional spaces, as well as with the evaluation and properties of integrals over appropriate lower dimensional domains. It accordingly includes such topics as the Gauss-Green theorem and Stokes' theorem, as well as matters related to the Plateau problem.

Concerning measures, perhaps the beginning of geometric measure theory was in Carathéodory's work on the length of a point set. This notion was extended by Hausdorff to measures of various dimensions, not necessarily numerical, but the basic idea is that of Carathéodory. A plethora of measures was soon in existence and much work was done especially in comparing them with each other. The most important among these measures are the Hausdorff measure and the integralgeometric measure. Much of this work may be only of academic interest. Besicovitch made a deeper study of lower dimensional measures. While his work was restricted to the one-dimensional measure of sets in the plane, it opened the possibility of studying the more general situation. The results deal with the relation between Hausdorff rectifiability of sets and their projection properties. A set of finite 1 measure is Hausdorff 1 rectifiable if all but a subset of Hausdorff 1 dimensional measure 0 lies on the union of countably many 1 manifolds of class one. It is a remarkable fact that a set of finite 1-measure is 1 rectifiable if and only if it has certain projection properties, i.e., compact subsets of the set which project on sets of measure 0 are of measure 0.

The methods of Besicovitch do not extend to the case of Hausdorff k dimensional measure in n space, $n>k$. Nevertheless, Federer was able to obtain the structure of sets of finite Hausdorff k measure in n space. This is an accomplishment of a very high order of magnitude. Although there have been fine contributions to the theory of Hausdorff and other measures, the paper by Federer "The (ϕ, k) rectifiable subsets of n space", published in 1947, is doubtless the high point of the theory.

Regarding the other aspect of geometric measure theory, i.e., the appropriate domains of integration, one should first mention the generalized curves and surfaces of L. C. Young. In the manner of distributions, defined later by Schwartz, and found by him and others to play a vital role in several branches of analysis, Young defined a surface by means of its action on a class of integrands. A second original and important contribution is the book of Whitney on "Geometric integration theory." The definitive work in this domain, however, is the paper "Normal and integral currents" by Federer and Fleming, published in 1961. I can do no better than to quote from the introduction. "Long has been the search for a satisfactory analytic and topological formulation of the concept 'k dimensional domain of integration in euclidean n space.' Such a notion must partake of the smoothness of differentiable manifolds and of the combinatorial structure of polyhedral chains with integer coefficients. In order to be useful for the calculus of variations, the class of all domains must have certain compactness properties. All the requirements are met by the integral currents studied in this paper." Later in the introduction, the authors say: "The basic techniques of this paper are measure theoretic, extended from ordinary functions to differential forms of arbitrary degree. The elegant linear functional approach to measure theory, which almost ignores that functions are defined on sets and have values at points, and that spheres are round, is followed throughout the first seven sections. However, in §8 it appears necessary to use more refined local techniques from the theory of relative differentiation of measures, a covering theorem of Vitali type, and the fundamental structure theorem from the theory of Hausdorff measure."

It would seem from the above remarks that, while the development of geometric measure theory is by no means a one man show, Federer is clearly the dominant figure in the building of this deep and important theory.

Federer's book gives a complete and self-contained treatment of geometric measure theory and its prerequisites. It starts at the beginning and goes to the frontier of the subject. Then it penetrates beyond the frontier and presents a good deal of important new material.

Chapter 1 is presented mainly for the sake of completeness. It furnishes the multilinear algebra needed for general purposes together with the particular aspects needed for geometric measure theory. Exterior algebra is treated as a Hopf algebra. In addition to the euclidean norms, the important norms, mass of an m vector and comass of an m covector, due to Whitney, are discussed. The definition of a normal current involves its mass and the mass of its boundary. Although multilinear algebra is used in so many branches of mathematics that it seems reasonable to assume it as part of the equipment of every graduate student, this is not the case at all, so that the inclusion of this preparatory material is essential.

Chapter 2 is independent of the remainder of the book. It may be used as a first course in measure theory or real variables. It is self contained and much of the material is rather standard. Some of the topics treated here in more detail than in other books are Suslin sets, which have recently become popular again, generalizations of the Besicovitch version of the Vitali covering theorem which, especially in a form due to A. P. Morse, has saved more than one mathematician from considerable heartache and many hours of labor, and a nice blending of abstract and special differentiation theory.

The last section of Chapter 2, Carathéodory's Construction, will be of special interest to many readers. There seems to be no other book which gives the definitions and elementary properties, except for cursory treatments, of the measures which come under this heading. For a metric space X, a family $\mathfrak{F}$ of subsets of X, and a function ζ on $\mathfrak{F}$, with $0 \leqq \zeta(S) \leqq \infty$, a measure is obtained in the following way. For each $0 < \delta \leqq \infty$, let ϕ_δ be defined by $\phi_\delta(A) = \inf \sum \zeta(S)$, where each summation is over a countable set in $\mathfrak{F}$ which covers A and each diam $S \leqq \delta$. Then $\psi(A) = \lim_{\delta \to 0} \phi_\delta(A)$. Special cases discussed are the Hausdorff measures and half a dozen or more others including the integralgeometric measure. Among the many interesting properties discussed in this section, I mention only the following result of Besicovitch. If $0 \leqq m < \infty$, $A \subset R^n$ is compact, $H^m(A) > 0$, then there is a compact $B \subset A$ with $0 < H^m(B) < \infty$. Davies and Rogers have recently given an example of a space and a Hausdorff type measure, not numerical dimensional, for which this result fails.

As I write this review, it occurs to me that it could be appropriate for each chapter to have a separate review. In that case, I think I would choose to be the reviewer of Chapter 3. If I were compelled to have a favorite chapter, I feel that this would have to be the one. The chapter is divided into four sections. The first section presents basic theorems on the differentiation of functions and may be considered to be a modernization of Saks. Accordingly, there are proofs that

(a) if $f: R^n \to R^m$ is Lipschitzian it is differentiable almost everywhere,

(b) it is approximately differentiable almost everywhere if and only if its approximate partial derivatives exist almost everywhere,

(c) it is differentiable at almost all points where

$$\limsup_{x \to a} \, |f(x) - f(a)| / |x - a| < \infty,$$

and

(d) it is approximately differentiable at almost all points where

$$\text{ap} \limsup_{x \to a} \, |f(x) - f(a)| / |x - a| < \infty.$$

The related theorem of Calderón and Serrin which says that for $f: R^n \to R^1$, f is differentiable almost everywhere if it is absolutely continuous and its partial derivatives are in L_p, $p > n$, is not given. The Whitney extension theorem is given a nice proof. The related theorem of Federer and Whitney which characterizes those mappings which are equal, except on sets of arbitrarily small measure, to mappings of class k, is proved. In this connection, H. E. White has characterized the one-one mappings which may be approximated in this way by diffeomorphisms. Among the other interesting facts in this section is the proof of the theorem, half due to Whitney and half new, that if $B\subset R^n$ is connected, and $k\geqq 1$, then B is a submanifold of class k of R^n if and only if there is a map of class k retracting some open subset of R^n onto B.

§2 deals with Lipschitzian maps of R^n into R^m, and especially with the area and coarea of such maps. For $f:R^m\to R^n$ Lipschitzian, $m\leqq n$, and $A\subset R^m$ measurable, the area formula

$$\int_A J_m f dL^m = \int_{R^n} N(f:A, y)dH^m$$

is obtained, where $J_mf(a)=\|\Lambda_m Df(a)\|$ is the Jacobian, and $N(f:A, y)$ is the cardinal number of the set $f^{-1}\{y\}\cap A$. The troublesome set where J is zero is handled in an ingenious new way. In this connection, Goffman and Ziemer have recently shown that if f is such that the partial derivatives of the coordinate mappings f_i are in L_{p_i}, with $p_i>m-1$ and $\sum (1/p_i)\leqq 1$ for every set of m distinct summands, then $\int J_m f dL^m = A(f)$, where A is the Lebesgue area. A transformation formula corresponding to the above area formula is given for Lipschitzian mappings. The coarea formula, discovered by Federer, refers to the case of a mapping $f:R^m\to R^n$, where $m>n$. It is obtained for Lipschitzian f and says that if $A\subset R^m$ is measurable then

$$\int_A J_n f(x) dL^m = \int_{R^n} H^{m-n}(A\cap f^{-1}\{y\})dL^n.$$

The example $n=1$, $f(x)=|x|$, is given in detail to show how this formula is used to obtain properties of the gamma function. A set is m rectifiable if it is the Lipschitzian image of a bounded set in R^m; it is (ϕ, m) rectifiable if its ϕ measure is finite and ϕ almost all of it is contained in the union of a countable number of m rectifiable sets; it is purely (ϕ, m) unrectifiable if no subset of positive ϕ measure is rectifiable. It is shown that (H^m, m) rectifiable sets have nice tangential properties. For (H^m, m) rectifiable sets the various measures agree; in particular, $H^m=I^m$, the integralgeometric measure. The (H^m, m) rectifiable sets, without the finite measure restriction, are those which, except for a subset of H^m measure zero, lie on the union of countably many m manifolds of class one. The following new result is obtained. If $W\subset R^n$ is (H^m, m) rectifiable and H^m measurable, $f: W\to R^n$ is Lipschitzian, k is an integer, $0\leqq k\leqq m$, $\lambda>0$, and

$$Z = [y: H^{m-k}(f^{-1}\{y\}) \geqq \lambda],$$

then Z is (H^k, k) rectifiable. Some integralgeometric formulas are obtained, the principal tool in their derivation being the coarea formula. Minkowski content is defined and is shown to be equal to Hausdorff measure for rectifiable sets. I have only given a sampling of the contents of this section.

The next section, entitled Structure Theory, is the high point of this chapter, and perhaps of the entire theory. Rectifiable sets are characterized in terms of their projection properties. Perhaps the most beautiful result is that if $A \subset R^n$ is a Borel set, $H^m(A) < \infty$, there is a Borel set $B \subset A$ such that $I^m(B) = H^m(B)$, $I^m(A \sim B) = 0$, and $A \sim B$ is purely (H^m, m) unrectifiable. This theorem, due to Federer, is presented with certain simplifications in the proof due to Mickle. The theorem is needed for the proof of the compactness theorem for integral currents given in Chapter 4. The section ends with a discussion of progress made on some still open matters regarding density and rectifiability.

The last section of Chapter 3 is in two parts. The first part deals with a matter which started with an interesting example discovered by Whitney of a mapping $f: R^2 \to R^1$ of class 1 whose set of singular points is carried into a set of positive measure. The situation was clarified by A. P. Morse who showed that if $f: R^n \to R^1$ is of class n then its set of singular points is mapped by f into a set of measure zero. Sard extended this theorem to mappings from R^n to R^m. Federer has now proved the optimal theorem of this sort. The proof, which depends in part on the arguments of Morse, is given here for the first time. If $m > \nu \geq 0$ and $k \geq 1$ are integers, $A \subset R^n$ is open, Y is a normed vector space, $f: A \to Y$ is of class k, and dim im $Df(x) \leq \nu$ for $x \in B$, then $H^{\nu + (m-\nu)/k}\,[f(B)] = 0$. Conversely, if $\gamma < \nu + (m - \nu)/k$ it can happen that $H^\gamma[f(B)] > 0$.

In the second part of the section, local algebraic and measure theoretic properties of sets defined by real analytic equations are derived. This section is highly algebraic. It is shown that a real analytic set has dimension m in the sense of local ideal theory if and only if its m dimensional Hausdorff measure is positive and locally finite.

Chapter 4 is an exposition of the work of Federer and Fleming, with many improvements, additions, and with new applications. Currents were introduced by deRham as objects which reflect the analysis and the topology of manifolds. It is accordingly natural that in seeking domains of definition of variational integrals one should consider special currents as likely candidates. In §1, the definitions and main properties of those currents which are important in geometric measure theory are presented. These are the normal, rectifiable, and integral currents. There is a short discussion of distributions in a general setting. Currents are a special case of distributions; for m currents the space of testing functions consists of m forms. The boundary ∂T of an m current T is an $m-1$ current defined by $\partial T(\phi) = T(d\phi)$. The mass $M(T)$ of a current is defined. The currents for which $M(T) < \infty$ are those which are representable by integration. Let $N(T) = M(T) + M(\partial T)$. T is called normal if $N(T) < \infty$ and T has compact support. Rectifiable currents are those which may be approximated, in a certain way, by sums of oriented simplexes. An integral current is one for which both T and ∂T are rectifiable. Flat chains are defined as certain limits of normal currents. Flat chains have important integralgeometric relationships which are established, e.g. if T is a flat chain, $m > 0$, then $I^m(\text{spt } T) = 0$ implies $T = 0$. Mapping properties of flat chains and rectifiable currents are given. New representations of n dimensional flat chains by means of pairs of L^n summable m and $m-1$ vector fields are given.

The main results of the general theory are exposed in §2. Let K be a compact Lipschitz neighborhood retract in R^n. The compactness theorem asserts that, for every nonnegative c, the set of integral currents with support in K and $N(T) \leqq c$ is compact in the metric given by the Whitney flat norm. The closure theorem asserts that the integral currents form a closed subset of the normal currents metrized by the flat norm. The section also includes the deformation theorem and approximation theorems of normal currents via polyhedral chains. A new proof of the closure theorem is given which applies to flat chains modulo ν, a theory developed by Ziemer and by Fleming.

§3 deals with slicing. Let U be an open set, let $f: U \to R^n$ be locally Lipschitzian, and let T be an m dimensional flat chain with compact support $K \subset U$, $m \geqq n$. An $m-n$ dimensional current $\langle T, f, y \rangle$, called the slice of T in $f^{-1}\{y\}$ is defined for almost every $y \in R^n$. Slices inherit the properties of the currents. This furnishes, in particular, a measure theoretic description of the algebraic geometric description of tangent cones at singular points for complex varieties. A generalization of the coarea formula is given.

In the next section, the author constructs the integral homology groups of local Lipschitz neighborhood retracts in R^n by using complexes of integral flat chains. The homology theory needed is all given in the book. The compactness properties of homology classes are established. These are used to obtain the existence of minimizing currents in Chapter 5.

The last section of this chapter deals with real functions on R^n which are locally summable and whose partial derivatives are representable by integration. Closely related are the sets of locally finite perimeter introduced by deGiorgi and given concrete meaning by Federer using the notion of restricted boundary. It is the set of points at which the Federer normal exists. It is equivalent to take it as the set of points at which both the set and its complement have nonzero upper density. The proof of this equivalence was published by Volpert. In terms of these notions a version of the Gauss-Green theorem which seems to be optimal is given. Stokes' theorem does not seem as yet to have been given such ultimate form, although I understand that Brothers has made recent progress on this issue.

The class of functions discussed in this section has a long history, and now seems to be the most appropriate extension to n dimensions of the functions of local bounded variation in 1-dimension. It now becomes clear for the first time that the theorem that right and left limits exist everywhere remains valid. The exceptional set, which is empty for $n=1$, is of Hausdorff $n-1$ measure zero, and limits are replaced by approximate limits. For $n=1$, bounded variation compels ordinary limits and approximate limits to be the same. A variety of other interesting properties of these functions, some of which are new, are obtained. The theorem of J. H. Michael on the approximation of functions whose partial derivatives are functions by means of functions of class one is not given.

The author's important work on surface area, using the methods of this chapter, is not discussed. Since the appearance of the book, the author has applied the methods of this chapter to obtain significant new results on the multiple Fourier integral.

The last chapter is concerned with minimizing problems for variational integrals. Based on work of Fleming and of Reifenberg, Almgren obtained general results on regularity of minimizing objects, somewhat like the generalized surfaces of L. C. Young. The machinery developed in Chapter 4 is now employed to give an exposition of this theory.

In §1, a parametric integrand Φ of degree m is considered and its integral over an m dimensional rectifiable current T is defined. Elliptic and semi-elliptic integrands are defined. If the classical parametric Legendre condition is satisfied then the integrand is elliptic. Lower semicontinuity is established with respect to the appropriate topology. The compactness results of Chapter 4 are used to deduce the existence of minimizing currents.

The second section has contact with the book by Morrey. Those parts of the theory of strongly elliptic systems of second order partial differential equations which are needed is expounded. Sobolev's in-