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NORMAL SOLVABILITY FOR NONLINEAR MAPPINGS INTO BANACH SPACES

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Let $X$ be a topological space, $Y$ a Banach space, $f$ a mapping of $X$ into $Y$. The mapping $f$ is said to be normally solvable (following a sort of terminology due to Hausdorff for linear operators) if its image $f(X)$ is closed in $Y$, with $Y$ given its strong topology. The objective of the theory of normally solvable mappings is to establish conclusions on the fine structure of the image set $f(X)$ from the hypothesis that $f(X)$ is closed in $Y$ together with hypotheses concerning the asymptotic direction set $D_x(f)$ of $f$ at various points $x$ of $f$, (conclusions which are also described as extensions of the Fredholm alternative to such nonlinear mappings $f$). The concept of asymptotic direction set is defined as follows:

**Definition 1.** Let $X$ be a topological space, $Y$ a Banach space, $f$ a mapping of $X$ into $Y$, $x$ a given point of $X$. Then the asymptotic direction set $D_x(f)$ of $f$ at $x$ is the subset of $Y$ defined by

$$D_x(f) = \bigcap_{\varepsilon > 0} \text{cl}(\{y \mid y \in Y, y = \xi(f(u) - f(x)), \xi \geq 0, u \in X, ||f(u) - f(x)|| < \varepsilon\}),$$

where $\text{cl}$ denotes the closure in the strong topology on $Y$.

Under sharper hypotheses, we have the following description of the asymptotic direction set:

**Proposition 1.** Let $X$ be a locally convex topological vector space, $Y$...
a Banach space, \( f \) a mapping of \( X \) into \( Y \) which is once Gateaux differentiable from \( X \) to \( Y \) at a given point \( x \) of \( X \) with differential \( df_x \) which is a continuous linear mapping from \( X \) to \( Y \). Let \( (df_x)^* \) be the dual mapping from \( Y^* \) to \( X^* \), \( N(df_x^*) \) its nullspace, and \( (N(df_x^*))^\perp \) its annihilator in \( Y \). Then:

\[
D_s(\mathcal{J}) \supset \text{cl}(df_x(X)) = (N(df_x^*))^\perp.
\]

Our basic result is the following:

**Theorem 1.** Let \( X \) be a topological space, \( Y \) a Banach space, \( f \) a mapping of \( X \) into \( Y \) such that \( f(X) \) is closed in \( Y \). Let \( y \) be a given point in \( Y \), and for \( r > 0 \), let \( B_r(y) \) be the closed ball of radius \( r \) about \( y \) in \( Y \). Suppose that there exists \( r > 0 \) and \( \rho < 1 \) such that \( f^{-1}(B_r(y)) \) is non-empty, while for each \( x \) in \( f^{-1}(B_r(y)) \),

\[
\text{dist}(y - f(x), D_s(f)) \leq \rho \|y - f(x)\|.
\]

Then: \( y \) lies in \( f(X) \).

A global analogue of Theorem 1 is the following:

**Theorem 2.** Let \( X \) be a topological space, \( Y \) a Banach space, \( f \) a mapping of \( X \) into \( Y \) such that \( f(X) \) is closed in \( Y \). Suppose that for each \( y \) in \( Y \), there exists \( r(y) > 0 \) and \( \rho(y) < 1 \) such that \( f^{-1}(B_{r(y)}(y)) \neq \emptyset \) for all \( y \) in \( Y \), while for each \( x \) in \( f^{-1}(B_{r(y)}(y)) \),

\[
\text{dist}(y - f(x), D_s(f)) \leq \rho(y) \|y - f(x)\|.
\]

Then: \( Y = f(X) \).

Using Proposition 1, we obtain the following specializations of these results:

**Corollary 1 to Theorem 1.** Let \( X \) be a locally convex topological vector space, \( Y \) a Banach space, \( f \) a once Gateaux differentiable mapping of \( X \) into \( Y \) with \( f(X) \) closed in \( Y \). Let \( y \) be a given element of \( Y \) and suppose for an \( r > 0 \) such that \( f^{-1}(B_r(y)) \neq \emptyset \) and for a given \( \rho < 1 \) that for all \( x \) in \( f^{-1}(B_r(y)) \), we have

\[
\|y - f(x) + N(df_x^*)^\perp \|_{Y/N(df_x^*)^\perp} \leq \rho \|y - f(x)\|.
\]

Then: \( y \) lies in \( f(X) \).

**Corollary 1 to Theorem 2.** Let \( X \) be a locally convex topological vector space, \( Y \) a Banach space, \( f \) a once Gateaux differentiable mapping of \( X \) into \( Y \) with \( f(X) \) closed in \( Y \). Suppose that the hypotheses of the Corollary 1 to Theorem 1 hold for each \( y \) in \( Y \). Then \( f(X) \) is the whole of \( Y \).
Specializing still further by taking \( p = 0 \) and \( p(y) = 0 \), respectively, we obtain the following:

**Corollary 2 to Theorem 1.** Let \( X \) be a locally convex topological vector space, \( Y \) a Banach space, \( f \) a once Gateaux differentiable mapping of \( X \) into \( Y \) with \( f(X) \) closed in \( Y \). Let \( y \) be an element of \( Y \), suppose that \( f^{-1}(B_r(y)) \neq \emptyset \) for a given \( r > 0 \). Suppose that for each \( x \) in \( f^{-1}(B_r(y)) \) and each \( y^* \) in \( N(df_x^*) \), we have

\[
(y^*, y - f(x)) = 0.
\]

Then: \( y \) lies in \( f(X) \).

**Corollary 2 to Theorem 2.** Let \( X \) be a locally convex topological vector space, \( Y \) a Banach space, \( f \) a once Gateaux differentiable mapping of \( X \) into \( Y \) such that \( f(X) \) is closed in \( Y \). Suppose that for each \( x \) in \( X \), \( N(df_x^*) = \{0\} \). Then: \( f(X) \) is the whole of \( Y \).

The special case of Corollary 2 to Theorem 1 in which \( Y \) is reflexive, \( f(X) \) is assumed to be weakly closed in \( Y \), and \( r = \text{dist}(y, f(X)) \) was given by Pohozhayev in [6]. The special case of Corollary 2 to Theorem 2 in which \( Y \) is uniformly convex was given by Pohozhayev [7]. The result of Theorem 2 for \( p(y) = 0 \) for all \( y \), which is roughly equivalent to assuming \( D_x(f) = Y \) for all \( x \) in \( X \), was given by the writer for general Banach spaces \( Y \) in Browder [3]. This was extended in Browder [4] to mappings into infinite dimensional manifolds \( Y \) with the condition on \( D_x(f) \) imposed upon \( x \) in \( X - N \) only, with the exceptional set \( N \) compact or satisfying other negligibility conditions. As we note from the above, Theorems 1 and 2 are considerably sharper and more general than the Corollaries 2 stated above.

We now proceed to the proof of Theorem 1, which is based upon the following Lemma:

**Lemma.** Let \( Y \) be a Banach space, \( S_0 \) a bounded closed subset of \( Y \), \( C \) a closed cone in \( Y \) generated by a closed bounded convex subset \( F \) of \( Y \) which does not contain \( 0 \). Then there exists an element \( s_0 \) of \( S_0 \) such that

\[
(s_0 + C) \cap S_0 = \{s_0\}.
\]

The proof of the Lemma is given in §1 of Browder [4] and is based on an extension of the idea of the proof of the Bishop-Phelps Theorem [1].

**Proof of Theorem 1.** Let \( S = f(X) \), and suppose that \( d_0 = \text{dist}(y, S) > 0 \). We shall deduce a contradiction. For a given \( \varepsilon > 0 \), which we shall specify later in the proof, we may choose a point \( s \) in \( S \) such that
\[ d = ||y-s|| \leq (1 + \varepsilon)d_0. \]

(If there exists a point \( s \) with \( ||y-s|| = d_0 \), we choose such an \( s \) in \( S \) and let \( \varepsilon = 0 \).) By hypothesis, there exists \( p < 1 \) such that for every \( x \) in \( f^{-1}(B_r(y)) \), there exists \( w \) in \( D_x(f) \) such that if \( \xi = ||y-f(x)|| \), then there exists \( w \) in \( D_x(f) \) such that \( ||\xi w - (y-f(x))|| \leq p\xi \) with \( 0 \leq p < 1 \).

We choose another constant \( q \) such that \( 0 \leq p < q < 1 \).

Let \( B \) be the closed ball of radius \( r = qd_0 \) about the point \( y \) in \( Y \). Let \( K \) be the convex closure of the union of the point \( \{ s \} \) and the ball \( B \).

Then \( K \) is a closed bounded convex subset of \( Y \), and \( u \) is any point of \( K \), \( u \) may be written in the form

\[ u = (1 - t)s + tz, \quad (z \in B, t \in [0, 1]). \]

Let \( S_0 = S \cap K \). Then \( S_0 \) is a closed bounded subset of \( Y \). If \( u \) lies in \( S_0 \), then

\[ d_0 \leq ||u - y|| \leq (1 - t)||s - y|| + t||z - y|| \leq (1 - t)(1 + \varepsilon)d_0 + tdq_0. \]

Hence

\[ (1) \quad t \leq \varepsilon(\varepsilon + (1 - q))^{-1}. \]

Let \( C \) be the closed cone with vertex at 0 in \( Y \) spanned by the closed bounded convex set \( F = (B-s) \) which does not contain 0. If we apply the Lemma to the set \( S_0 \) and the cone \( C \), it follows that there exists a point \( s_0 \) in \( S_0 \) such that \( (s_0 + C) \cap S_0 = \{ s_0 \} \). Since \( s_0 \) lies in \( S_0 \), \( s_0 = (1 - t)s + tz \), with \( z \) in \( B \) and \( t \) in \([0, 1]\) satisfying the inequality \( (1) \) above. If \( y \) is an element of \( C \), \( y \) can be written in the form

\[ y = \xi(z_1 - s), \quad (\xi \geq 0, z_1 \in B). \]

Suppose that \( y \neq 0 \), and that \( v = (s_0 + y) \) lies in \( S \). Then:

\[ v = (1 - t)s + tz + \xi(z_1 - s) = (1 - t - \xi)s + tz + \xi z_1 \]

\[ = (1 - (t + \xi))s + (t + \xi)[t(t + \xi)^{-1}z + \xi(t + \xi)^{-1}z_1]. \]

Suppose that

\[ \xi \leq (1 - q)(\varepsilon + (1 - q))^{-1} = \delta, \quad (\delta > 0). \]

Then \( (t + \xi) \leq t + \delta \leq 1 \), and \( v \) lies in \( K \). Then we should have \( v \) in \( S \cap K = S_0 \), which contradicts the fact that \( S_0 \cap (s_0 + C) = \{ s_0 \} \). Hence for any such \( v \), \( \xi > \delta \), so that we have \( ||y|| = ||\xi||z_1 - s|| > \delta(1 - q)d_0 = \delta t \).

Thus,

\[ (s_0 + C) \cap S \cap B_{\delta t}(s_0) = \{ s_0 \}. \]

Hence, for any point \( x \) in \( X \) for which \( s_0 = f(x) \), it follows that

\[ D_x(f) \cap \text{Int}(C) = \emptyset. \]
where $\text{Int}(C)$ denotes the interior of the cone $C$ in $Y$. For any such point $x$, we have

$$\|y - f(x)\| \leq \|y - s\| + \|s - s_0\|,$$

with $(s-s_0) = t(s-z)$ in terms of the representation for $s_0$ considered above with $z$ in $B$. Therefore,

$$\|y - f(x)\| \leq (1 + \epsilon)d_0 + \epsilon(\epsilon + (1 - q))^{-1}(1 + \epsilon + q)d_0 = d_0 + \epsilon sd_0.$$

If the constant $r$ of the hypothesis exceeds $d_0$, we may choose $\epsilon > 0$ so small that $d_0 + \epsilon sd_0 \leq r$. If $r = d_0$, we choose $\epsilon = 0$, $s = s_0$, and $x$ automatically lies in $f^{-1}(B_r(y))$. In both cases, we may choose $\epsilon$ sufficiently small so that $x$ lies in $f^{-1}(B_r(y))$.

Finally we conclude the proof by deducing that $D_x(f) \cap \text{Int}(C)$ is nonempty for small $\epsilon$ which contradicts our preceding argument. For the given point $x$, there exists $w$ in $D_x(f)$ such that for $\xi = \|y-f(x)\| = \|y-s_0\|$, we have

$$\|\xi w - (y - s_0)\| \leq p\xi,$$

i.e.

$$\|(s_0 + \xi w) - y\| \leq p\|y - s_0\| \leq pd_0 + \epsilon psd_0.$$

We choose $\epsilon$ so small that $p + \epsilon ps < q$. Then $(s_0 + \xi w)$ lies in the interior of the ball $B$, i.e. $\xi w$ lies in the interior of $(B - s_0)$. Hence $\xi w$ lies in the interior of $C$, and so does $w$ itself, i.e. $w \in D_x(f) \cap \text{Int}(C)$.

This contradiction to the initial assumption that $d_0$ is positive establishes the validity of the theorem. q.e.d.

**BIBLIOGRAPHY**


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