A SHORT PROOF OF A THEOREM OF PLANS ON
THE HOMOLOGY OF THE BRANCHED CYCLIC
COVERINGS OF A KNOT

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Let $K \subset S^3$ be a (tame) knot, with complement $C = S^3 - K$, and let $\bar{C}$ be the infinite cyclic covering of $K$, i.e. the covering of $C$ corresponding to the commutator subgroup of $\pi_1(C)$. The group of covering translations of $\bar{C}$ is $H_1(\bar{C})$, which is infinite cyclic by Alexander duality; this gives an action of $\mathbb{Z}$ on $H_1(\bar{C})$, and so $H_1(\bar{C})$ becomes a $\Lambda$-module, where $\Lambda$ is the integral group ring of $\mathbb{Z}$. We identify $\Lambda$ with the ring of polynomials in a single variable $t$, (positive and negative powers of $t$ being allowed), with integral coefficients. (See [4].)

The $k$-fold branched cyclic covering of $K$, $M_k$ ($k \geq 1$) is defined by taking the covering of $C$ corresponding to the kernel of the composition:

$$\pi_1(C) \rightarrow H_1(C) \cong \mathbb{Z} \rightarrow \mathbb{Z}_k,$$

and branching about $K$. (For more details, see [1], [4].) $M_k$ is a closed, orientable 3-manifold; for example, $M_1$ is just $S^3$.

If $M(t) = (m_{ij}(t))$, $m_{ij}(t) \in \Lambda$, is a presentation matrix for $H_1(\bar{C})$ as a $\Lambda$-module, then it can be shown that a presentation matrix for $H_1(M_k)$ ($k > 1$) as an abelian group is obtained by substituting for each entry $m_{ij}(t)$, which is some finite formal sum, $\sum_{r} \alpha_i t^r$, say, the $k \times k$ block $\sum_{r} \alpha_i T^r_k$, where the summation indicates ordinary matrix addition, and $T^r_k$ is the $k \times k$ matrix:

$$\begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
1 & 0 & 0 & \cdots & 0
\end{pmatrix}.$$
(and $T_k$ is defined to be the $k \times k$ identity matrix). (See [2], [4].) Call the matrix obtained from $M(t)$ in this way $M(T_k)$.

Now a geometrical description of $C$, in terms of an orientable surface spanning $K$, shows that we may take $M(t)$ to be of the form $tV - VT^T$, where $V$ is a $2h \times 2h$ matrix over $\mathbb{Z}$ ($h \geq 1$, the genus of $K$) and $VT^T$ is the transpose of $V$. (See [1], [6].) $M(T_k)$ is then $2hk \times 2hk$, but Seifert showed (see [1], [6]) that it is in fact equivalent (in the sense of presenting the same abelian group) to a $2h \times 2h$ matrix, $F_k$ say. In [5], Plans shows that this $F_k$ can be expressed in terms of two matrices $P_k$ and $Q_k$, which in turn are defined by certain recurrence relations analogous to those defining the Fibonacci numbers. These facts are used to effect a diagonalisation of $F_k$, from which some interesting general conclusions about $H_1(M_k)$ are drawn, perhaps the most striking being the following:

**Theorem (Plans).** If $k$ is odd, then $H_1(M_k)$ is a direct double, i.e. $H_1(M_k) \cong G \oplus G$, for some $G$.

The proof given in [5] is rather long and involved, and it is the purpose of this note to show that, although $F_k$ is smaller than $M(T_k)$, the above result actually follows very easily from an examination of the big matrix.

**Proof.** $M(t) = tV - VT^T$. So by a suitable sequence of row interchanges and column interchanges, $M(T_k)$ can be brought into the form:

$$
\begin{pmatrix}
-V^T & V & 0 & 0 & \cdots & 0 & 0 \\
0 & -V^T & V & 0 & \cdots & 0 & 0 \\
0 & 0 & -V^T & V & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & -V^T & V \\
V & 0 & 0 & 0 & \cdots & 0 & -V^T
\end{pmatrix}
$$

a $k \times k$ matrix of $2h \times 2h$ blocks.

Now if $k$ is odd, $k = 2r + 1$ say, a further sequence of row interchanges gives a matrix in which the rows of blocks occur in the order (numbering them according to their positions in the old matrix): $r + 1, r + 2, \cdots, 2r + 1, 1, 2, \cdots, r$. It is easy to see that this new matrix is skew-symmetric. We illustrate the case $k = 7$:
But it is well known (see for example [3, p. 52]) that any skew-symmetric $2n \times 2n$ matrix over $\mathbb{Z}$ is equivalent to a block diagonal matrix of the form:

$$
\begin{pmatrix}
0 & 0 & 0 & -V^T & V & 0 & 0 \\
0 & 0 & 0 & 0 & -V^T & V & 0 \\
0 & 0 & 0 & 0 & 0 & -V^T & V \\
V & 0 & 0 & 0 & 0 & 0 & -V^T \\
-V^T & V & 0 & 0 & 0 & 0 & 0 \\
0 & -V^T & V & 0 & 0 & 0 & 0 \\
0 & 0 & -V^T & V & 0 & 0 & 0
\end{pmatrix}
$$

and hence presents a direct double.

**REFERENCES**


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