DEFORMATIONS OF LIE SUBGROUPS

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1. Introduction. This note is an announcement of results concerning the local deformation theory of subgroups of a Lie group. Let \( G \) be a real (resp. complex) Lie group and let \( M \) be a real (resp. complex)-analytic manifold. Roughly speaking, an analytic family of Lie subgroups of \( G \), parametrized by \( M \), is an analytic submanifold \( \mathfrak{X} \) of \( G \times M \) such that each "fibre" \( H_t \) is a Lie subgroup of \( G \); here the "fibre" \( H_t \) is defined by \( \mathfrak{X} \cap (G \times \{t\}) = H_t \times \{t\} \). (See §2 for a precise definition of an analytic family of Lie subgroups.) Our main result concerning such families is

**Theorem A.** Let \( \mathfrak{X} = (H_t)_{t \in M} \) be an analytic family of Lie subgroups of \( G \), let \( t_0 \in M \) and let \( H = H_{t_0} \). Let \( K \) be a Lie subgroup of \( H \) such that the component group \( K/K^0 \) is finitely generated and such that the Lie group cohomology space \( H^1(K, \mathfrak{g}/\mathfrak{h}) \) vanishes. Then there exists an open neighborhood \( U \) of \( t_0 \) in \( M \) and an analytic map \( \beta : U \rightarrow G \) such that \( KS_0 \beta(t)H_0 \beta(t)^{-1} \) for every \( t \in U \).

Here \( \mathfrak{g} \) (resp. \( \mathfrak{h} \)) denotes the Lie algebra of \( G \) (resp. \( H \)) and the \( K \)-module structure of \( \mathfrak{g}/\mathfrak{h} \) is determined by the adjoint representation of \( K \) on \( \mathfrak{g} \).

Theorem A generalizes the result of A. Weil [6, p. 152] which states that if \( \Gamma \) is a discrete, finitely generated subgroup of \( G \) such that \( H^1(\Gamma, \mathfrak{g}) = 0 \), then \( \Gamma \) is "rigid". It also generalizes results of the author [4], [5] on deformations of subalgebras of Lie algebras to the case of Lie subgroups. The proof of Theorem A depends heavily on the analyticity assumptions, although we suspect that the \( C^\infty \) version of the theorem is also true.

If \( G \) acts as an analytic transformation group on the analytic manifold \( M \) and if all orbits of \( G \) on \( M \) have the same dimension, then it can be shown that the connected isotropy groups \( (G_{0}^0)_{t \in M} \) form an analytic family of Lie subgroups of \( G \), and hence Theorem A applies. For example, let \( K \) be a maximal compact subgroup of \( G_{0}^0 \). Then

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$H^1(K, g/\mathfrak{g}_t) = 0$ and thus there exists a neighborhood $U$ of $t$ such that $G^s_t$ contains a subgroup conjugate to $K$ for every $s \in U$. For the case of algebraic transformation groups (over $C$) one gets considerably stronger theorems along the same line.

2. Analytic families of Lie subgroups. Analytic manifolds and Lie groups are taken over either the field $R$ of real numbers or the field $C$ of complex numbers. Analytic submanifolds and Lie subgroups are defined as in [2]; in particular analytic submanifolds and Lie subgroups are not required to have the topology induced by the ambient manifold. The Lie algebra of a Lie group $G$ will be denoted by the corresponding German letter $\mathfrak{g}$ and the connected component of the identity in $G$ will be denoted by $G^0$.

Definition 2.1. Let $G$ be a Lie group and let $M$ be an analytic manifold. Then an analytic family of Lie subgroups of $G$, parametrized by $M$, is an analytic submanifold $\mathcal{X}$ of $G \times M$ which satisfies the following conditions:

(a) Let $\pi_M: \mathcal{X} \to M$ denote the composition $\text{pr}_M \circ i$, where $i: \mathcal{X} \to G \times M$ is the inclusion map and $\text{pr}_M$ is the projection $G \times M \to M$. Then $\pi_M$ is surjective and is a submersion.

(b) Each fibre $\pi_M^{-1}(t)$ ($t \in M$) is of the form $H_t \times \{t\}$, where $H_t$ is a Lie subgroup of $G$.

(c) Let $\mathcal{X} \times_M \mathcal{X} = \{(a, b) \in \mathcal{X} \times \mathcal{X} \mid \pi_M(a) = \pi_M(b)\}$ and let $m: \mathcal{X} \times_M \mathcal{X} \to \mathcal{X}$ be defined by $m((x, t), (y, t)) = (xy, t)$ and $s(x, t) = (x^{-1}, t)$. Then $m$ and $s$ are analytic maps.

It follows from the definition that the function $t \to \text{dim } H_t$ is constant on each component of $M$.

3. Sketch of the proof of Theorem A. Let $F$ denote either $R$ or $C$. Since the problem is local, we may assume that $M$ is an open neighborhood of $0$ in $F^r$ and that $t_0 = 0$. We let $W$ be a vector subspace of $\mathfrak{g}$ which is complementary to $\mathfrak{h}$. If $\pi: \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is the canonical projection, then the restriction $\pi_W$ of $\pi$ to $W$ is a vector space isomorphism of $W$ and $\mathfrak{g}/\mathfrak{h}$; we define a $K$-module structure on $W$ by transferring the $K$-module structure on $\mathfrak{g}/\mathfrak{h}$ to $W$ by means of $\pi_W$. Let $\eta: K \to \text{GL}(W)$ denote the corresponding representation.

The following lemma is proved by a straightforward application of the implicit function theorem.

Lemma 3.1. There exists an open neighborhood $U$ of $H \times \{0\}$ in $H \times M$ and an analytic map $\psi: U \to W$ such that the following conditions hold for all $(x, t) \in U$:

(a) $\psi(x, 0) = 0$;

(b) $(\exp \psi(x, t))x \in H_t$;
(c) the map \((x, t) \mapsto (\exp \psi(x, t)x, t)\) is an analytic diffeomorphism of \(U\) onto an open neighborhood of \(H \times \{0\}\) in \(\mathcal{C}\).

The function \(\psi\) is called the normal displacement function of the family \(\mathcal{C}\). The germ of \(\psi\) in a neighborhood of \(H \times \{0\}\) is uniquely determined by the family \(\mathcal{C}\).

For each \(x \in K\), the map \(\psi_x: t \mapsto \psi(x, t)\) is an analytic map of an open neighborhood \(U_x\) of 0 in \(M\) into \(W\). Thus we may expand \(\psi_x\) in a convergent power series about 0,

\[
\psi_x(t) = \sum_{m=1}^{\infty} P_m(x, t),
\]

where, for each \(m\), \(t \mapsto P_m(x, t)\) is the restriction to \(U_x\) of a homogeneous polynomial map of degree \(m\) of \(F^r\) into \(W\); denote this homogeneous polynomial map by \(Q_m(x)\). If \(\mathcal{O}_m\) denotes the vector space of all homogeneous polynomial maps of \(F^r\) into \(W\), then \(Q_m: K \to \mathcal{O}_m\) is an analytic map. Let \(s\) denote the smallest integer \(j\) such that \(Q_j \neq 0\). \(Q_s\) is called the first nonvanishing infinitesimal displacement along \(K\) of the analytic family \(\mathcal{C}\).

We define a \(K\)-module structure on \(\mathcal{O}_m\) as follows: If \(x \in K\) and \(Q \in \mathcal{O}_m\), then \(x \cdot Q = \eta(x) \circ Q\). It follows easily from the hypothesis that \(H^1(K, \mathcal{O}_m) = 0\).

**Proposition 3.2.** \(Q_s\) is a one cocycle of \(K\).

Since \(H(K, \mathcal{O}_s) = 0\), it follows from Proposition 2.2 that there exists \(\phi_s \in \mathcal{O}_s\) such that \(P_s(x, t) = \phi_s(t) - x \cdot \phi_s(t)\) for \(x \in K\) and \(t \in U_x\). Using this, it can be shown that if we replace the analytic family \(\mathcal{C}\) by the family

\[
\mathcal{C}' = (\exp \phi_s(t))H_t(\exp - \phi_s(t))_{t \in M},
\]

then the first nonvanishing infinitesimal displacement along \(K\) of the analytic family \(\mathcal{C}'\) is of degree \(\geq s+1\).

Continuing inductively, we can define an infinite family \((\phi_u)_{u=s, s+1, \ldots}\), where \(\phi_u\) is a homogeneous polynomial map of \(F^r\) into \(W\) of degree \(u\), such that the following condition holds: let \(u \geq s\), let \(\phi^u = \phi_s + \phi_{s+1} + \cdots + \phi_u\) and let \(\mathcal{C}_u\) denote the analytic family \((\exp \phi^u(t))H_t(\exp - \phi^u(t))_{t \in M}\); then the first nonvanishing infinitesimal displacement of the family \(\mathcal{C}_u\) is of degree greater than \(u\).

Let \(\phi\) denote the formal power series map of \(M\) into \(W\) given by \(\phi = \phi_s + \phi_{s+1} + \cdots\). If \(\phi\) converges in a neighborhood of 0 and if \(\beta(t) = \exp \phi(t)\) then it is easy to see that \(\beta\) satisfies the conditions of Theorem A. At this point, we need to use a recent theorem of M.
Artin [1]. Very roughly, Artin’s theorem says that if a finite number of analytic equations admit a formal power series solution, then they admit a convergent power series solution. With some work, we can show that Artin’s theorem implies that the formal power series $\phi$ above can be chosen to be convergent, which proves Theorem A.

4. Applications to analytic transformation groups. Let the Lie group $G$ act as an analytic transformation group on the analytic manifold $M$. If $t \in M$, then the subgroup $G_t = \{ g \in G \mid g \cdot t = t \}$ is called the isotropy group of $G$ at $t$; the identity component $G_0$ is the connected isotropy group at $t$.

**Proposition 4.1.** Let $G$ act on $M$ as above and assume that all orbits of $G$ on $M$ have the same dimension. Then the family of connected isotropy groups $(G_0^t)_{t \in M}$ is an analytic family of Lie subgroups of $G$.

Thus we see that Theorem A applies to the situation above. A Lie group $G$ is reductive if the component group $G/G^0$ is finite, if $G$ admits a faithful finite-dimensional analytic representation and if every finite-dimensional analytic representation of $G$ is completely reducible. If $G$ is reductive and if $\rho: G \rightarrow \text{GL}(V)$ is an analytic representation of $G$, then it is easy to see that $H^1(G, V) = 0$.

Now let $G$ and $M$ be as in Proposition 4.1, let $t \in M$ and let $K$ be a reductive subgroup of $G_0^t$. Then Theorem A implies that there exists a neighborhood $U$ of $t$ on $M$ such that $G_0^t$ contains a conjugate of $K$ for every $y \in U$.

5. Applications to algebraic transformation groups. Let $G$ be a complex linear algebraic group and let $G$ act as an algebraic transformation group on the complex algebraic variety $M$.

**Proposition 5.1.** There exists a nonempty, Zariski-open subset $U$ of $M$ such that the family $(G_t)_{t \in U}$ is an analytic family of Lie subgroups of $G$.

If $S$ is a complex linear algebraic group, then it is known (see [3]) that $S$ admits a semidirect decomposition $S = R \cdot U$, where $U$ is the unipotent radical of $S$ and $R$ is a reductive algebraic subgroup of $S$; $R$ is determined to within conjugacy by elements of $U$. Such a decomposition is called a Levi decomposition of $S$.

Now let $(G, M)$ be an algebraic transformation space as above and, for every $t \in M$, let $G_t = R^t \cdot U_t$ be a Levi decomposition of $G_t$. Then the following theorem is a consequence of Theorem A and Proposition 5.1.
Theorem B. There exists a finite family $X_1, \ldots, X_n$ of Zariski-locally closed subsets of $M$ such that the following conditions hold:

(a) $M = \bigcup_{i=1}^{n} X_i$.
(b) For each $j$, $X_j$ is a Zariski-open subset of $M - \bigcup_{i=1}^{j} X_i$.
(c) If $x, y \in X_j$, then $R_x$ and $R_y$ are conjugate.
(d) For each $j$, the family $(U_i)_{i \in X_j}$ is an analytic family of Lie subgroups of $G$.

References