KOEBE SETS FOR UNIVALENT FUNCTIONS WITH TWO PREASSIGNED VALUES

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1. Introduction. Let \( \mathcal{M}_M \) denote the set of all functions \( f(z) \) that are analytic and univalent in the unit disc \( \Delta \) and satisfy the conditions

\[
\begin{align*}
  f(0) &= 0, \quad f(z_0) = z_0, \quad |f(z)| \leq M, \\
  &\text{where } z_0 \text{ is a fixed point of } \Delta, \\
  &z_0 \neq 0, \quad \text{and where } M \text{ is fixed, } 1 < M \leq \infty.
\end{align*}
\]

Although the class \( \mathcal{M}_M \) has been a popular one to study, very little seems to have been done with \( \mathcal{M}_M \). We aim to correct this oversight by beginning a study of \( \mathcal{M}_M \). In this paper we obtain the exact value of the "Koebe constant" for \( \mathcal{M}_M \) and we determine the Koebe sets for

(i) the set \( \mathcal{M}_M^* \) consisting of those elements \( f(z) \) of \( \mathcal{M}_M \) for which \( f(\Delta) \) is starlike with respect to the origin, and

(ii) the set \( \mathcal{M}_M^{\infty} \) consisting of those members \( f(z) \) of \( \mathcal{M}_M \) for which \( f(\Delta) \) is convex in the direction \( e^{ia} \).

2. Main results. By the Koebe constant for \( \mathcal{M}_M \) we mean the radius of the largest disc, center at the origin, that lies in the set

\[
\bigcap \{ f(\Delta) \mid f \in \mathcal{M}_M \},
\]

the Koebe set for \( \mathcal{M}_M \).

THEOREM 1. The Koebe constant for \( \mathcal{M}_M \) is given by

\[
\frac{2^{s^2} - 2s(\delta^2 - M)}{2^{s^2} - 2s - M} = \frac{M - \|z_0\|}{1 - \|z_0\|}.
\]

This result is sharp.

Proof. First, there is no loss of generality here if \( z_0 \) is taken to be real and positive. Hence we set \( z_0 = r_0 > 0 \). Now we obtain the domain \( \Omega^* \) from the domain \( \Omega = f(\Delta) \) by a circular symmetrization with respect to the half-line \([0, r_0, \infty)\). The domain \( \Omega^* \) contains the origin

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and it contains the point \( r_0 \); moreover, it is contained in a domain \( D_h \) which is the disc \( |w| < M \) slit along the segment \((-M, -h)\).

Well-known monotonic properties of the hyperbolic distance give us the inequalities

\[
\text{arc tgh } r_0 = \rho(0, r_0, \Omega) \geq \rho(0, r_0, \Omega^*)
\]

\[
\geq \rho(0, r_0, D_h) = \text{arc tgh } |\phi(r_0)|,
\]

where \( w = \phi(z) \) is a function that maps \( \Delta \) conformally onto \( D_h \) subject to the condition \( \phi(0) = 0 \). A computation involving \( \phi(z) \) shows that (2) holds if (1) holds, with \( z_0 = r_0 \). The equality sign in (1) holds for the function \( f(z) \) defined by

\[
\frac{f(z)}{[M - e^{-\alpha f(z)}]^2} = \left(1 - \left| \frac{z}{z_0} \right| \right)^2 \frac{z}{(1 - e^{-\alpha z})^2}, \quad z_0 = r_0 e^{i\alpha}.
\]

This completes the proof of Theorem 1.

**Remark.** For \( M = \infty \), (1) gives us a result due to Lewandowski [3], and for \( z_0 = 0 \), (1) yields a result due to Pick [5].

Now we shall determine the Koebe set for \( \mathfrak{M}_M^* \), the set of elements of \( \mathfrak{M}_M \) that map \( \Delta \) onto domains that are starlike with respect to the origin.

First we recall some facts about Koebe sets. If \( \mathfrak{E} \) is a nonempty set of functions \( f(z) \), analytic in \( \Delta \), then the **Koebe set** of \( \mathfrak{E} \) is the set

\[
\mathfrak{K}(\mathfrak{E}) = \bigcap \{ f(\Delta) \mid \mathfrak{E} \}, \quad [1].
\]

However, for the set \( \mathfrak{M}_M^* \), Krzyż and Zlotkiewicz found another characterization of \( \mathfrak{K}(\mathfrak{M}_M^*) \). Let \( \Delta_M \) denote the disc \( |w| < M \) and let \( \mathcal{G} \) denote the set of all subdomains \( D \) of \( \Delta_M \) that (i) are starshaped with respect to the origin, and (ii) contain the fixed point \( z_0 \). For \( D \in \mathcal{G} \), let \( g(w, z_0, D) \) be the Green’s function with pole at \( z_0 \) and let \( \mu(w_0) \) be defined by

\[
\mu(w_0) = \text{lub} \{ g(0, z_0, D) \mid D \in \mathcal{G}, w_0 \in \Delta_M \setminus D \}.
\]

Then the result due to Krzyż and Zlotkiewicz, alluded to above, is that

\[
\mathfrak{K}(\mathfrak{M}_M^*) = \left\{ w \mid \mu(w) < \log \frac{1}{|z_0|} \right\}
\]

holds [2].

Now if we make use of (3) and (4), then we obtain the following result.

**Theorem 2.** The set \( \mathfrak{K}(\mathfrak{M}_M^*) \) is determined by the condition

\[
|w - w_0| \left| \frac{M^2 - w_0^2}{M + |w|} \right| + \frac{1}{|w|} \left| \frac{Mw + z_0}{M + |w|} \right| < \frac{1}{2} (1 + |z_0|^2),
\]
and the Koebe set $\mathcal{K}(\mathfrak{M}_M^*)$ is determined by the inequality

$$(5) \quad |w - z_0| + |w| < \frac{1}{2}(1 + |z_0|^2).$$

**Remark.** The elliptic domain (5) is a well-known one due to Rogosinski [6].

The formula in (4) can be applied to other subclasses of $\mathfrak{M}_M$. For example, we have found the following result.

**Theorem 3.** The Koebe set $\mathcal{K}(\mathfrak{M}_M^*)$ is determined by the inequality

$$1 + [A(1 + \cos 2\theta) + (B^2 - 1) \cos 2\theta + BC \sin 2\theta]^{1/2} < \frac{1 - (1 - D^2)^{1/2}}{|z_0|^2},$$

where

$$h = |w|, \quad d = |w - z_0|, \quad \theta = \alpha - \arg z_0,$$

$$A = \frac{2hd}{(h + d)^2}, \quad B = \frac{h - d}{|z_0|}, \quad D = \frac{|z_0|}{h + d},$$

$$C = [(1 - D^2)(2A + D^2 - 1)]^{1/2}.$$

The set $\mathcal{K}(\mathfrak{M}_M^*)$ is a simply-connected Jordan domain if $|z_0| [1 + |\sin \theta|]^{1/2} < 1, \theta \neq 0, \pi$, while $\mathcal{K}(\mathfrak{M}_M^*)$ is the union of two disjoint simply-connected Jordan domains, which are symmetric with respect to the point $\frac{1}{2}z_0$, if $1 < |z_0| [1 + |\sin \theta|]^{1/2}, \theta \neq 0, \pi$.

**Remark.** For $z_0 = 0$ we obtain

$$\mathcal{K}(\mathfrak{M}_M^{*2}) = [w \mid s |w| (|w| + |\text{Im } w|) < 1],$$

which is related to a result due to McGregor [4].

**References**


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