HOMOTOPY THEORY OF RINGS AND ALGEBRAIC K-THEORY

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ABSTRACT. Algebraic $K$-theory is interpreted in terms of standard homotopy notions applied to the category of rings. Representability of the functors $K^{−i}$ is discussed.

The object of this announcement is to indicate how the algebraic $K$-theory of [1] and [2] can be explained as homotopy theory in a precise sense. Some terminology of homotopy theory was used in both these articles, but the analogy turns out to be very far reaching. We work in the category $𝒜$ of Banach rings complete in their quasi-norm [2]; morphisms are bounded homomorphisms. The terminology of [2] will be assumed. From $𝒜$ one constructs the category Hot-$𝒜$ whose objects are those of $𝒜$ and morphisms are homotopy classes of bounded maps (some care must be observed in defining Hot-$𝒜$ since homotopy is per se neither transitive nor symmetric but does behave well with respect to compositions). Denote Hot-$𝒜(A, B)$ by $[A, B]$.

DEFINITION 1. If $X$ and $A \to B$ are in $𝒜$, one says that $f$ is an $X$-fibration if for each $n \geq 1$, $E^n f$ induces a surjection $𝒜(X, E^n A) \to 𝒜(X, E^n B)$.

LEMMA 1. For all $X$ in $𝒜$, $A \{x \to A$ and $E A \to A$ given by “$x \to 1$” are $X$-fibrations.

DEFINITION 2. The mapping cone $C(g)$ of $g: B \to C$ is the fibre product in the diagram

$$
\begin{array}{ccc}
C(g) & \longrightarrow & EB \\
\downarrow g_1 & & \downarrow \\
A & \xrightarrow{g} & B
\end{array}
$$

LEMMA 2. For any $X$ there is an exact sequence of pointed sets

$$[X, C(g)] \to [X, A] \to [X, B].$$


Key words and phrases. Algebraic $K$-theory, homotopy in rings, Puppe sequences, representable functor, pro-rings.

1 I believe a more appropriate terminology for $E A$ and $\Omega A$ would have been the cone and suspension of $A$ respectively.

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One may iterate the construction $C(g)$ to get the diagram

$$\cdots \rightarrow C(g_n) \xrightarrow{g_{n+1}} C(g_{n-1}) \rightarrow \cdots \rightarrow C(g) \xrightarrow{g_1} A \xrightarrow{g} B$$

and the corresponding exact Puppe sequence of homotopy sets

$$\rightarrow [X, C(g_n)] \rightarrow [X, C(g_{n-1})] \rightarrow \cdots \rightarrow [X, A] \rightarrow [X, B].$$

Assume now that $X$ is a cogroup in $\mathcal{B}$. That is, there is a morphism $X \rightarrow X \sqcup X$ in $\mathcal{B}$ such that $(1_X \sqcup 0) \cdot \Delta = 1_X$, $(0 \sqcup 1_X) \circ \Delta = 1_X$ and $(\Delta \sqcup 1_X) \circ \Delta = (1_X \sqcup \Delta) \circ \Delta$. Then $\mathcal{B}(X, A)$ is a group for all $A$.

**Proposition 1.** If $X$ is a cogroup in $\mathcal{B}$, then the functor $\mathcal{B} = \mathcal{B}(X, \cdot) : \mathcal{B} \rightarrow \text{Groups}$ is a Mayer-Vietoris Functor in the sense of [1]. In addition $[X, A] = \kappa_1(A)$, where in the terminology of [1] (using the appropriate path ring $EA = xA \{x\} \kappa_1(EA)$ is defined by the exact sequence

$$X(EA) \rightarrow X(A) \rightarrow \kappa_1(A) \rightarrow 1.$$

**Theorem 1.** Assume again that $X$ is a cogroup in $\mathcal{B}$ and the diagram

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is a short exact sequence in $\mathcal{B}$ with $g$ an $X$-fibration. Then the exact Puppe sequence above is precisely the exact K-theory sequence of [1]

$$\cdots \rightarrow \kappa_{n+1}(A) \rightarrow \kappa_{n+1}(B) \rightarrow \kappa_{n+1}(C) \rightarrow \kappa_n(A) \rightarrow \cdots.$$

(Again, $EA$ must be suitably interpreted in [1] for nondiscrete rings.) We may apply these notions to the functors $G_{\lambda n}$ and $G\lambda$.

**Proposition 2.** $G_{\lambda n}$ is representable by $g_{\lambda n}$ in $\mathcal{B}$. $G\lambda$ is pro-representable in $\mathcal{B}$ by $g\lambda$. Both $g_{\lambda n}$ and $g\lambda$ are cogroups, the latter in pro-$\mathcal{B}$.

**Corollary.** For any $A$ in $\mathcal{B}$ we have canonical isomorphisms $\kappa_{g_{\lambda n}}(A) \cong [g_{\lambda n}, A]$ and $K^{-1}(A) = \kappa_{g\lambda}(A) = [g\lambda, A]$, where the last equation is interpreted in the pro-homotopy category. Furthermore, the exact Puppe sequence for $X = g\lambda$ in Theorem 1 is precisely the exact sequence of [2] of the functors $K^{-n}$.

We can also consider the representability of the functors $K^{-n}$ of [2].

**Lemma 3.** The loop ring functor $A \rightarrow \Omega A$ has an adjoint $\Sigma$ in pro-$\mathcal{B}$. One has pro-$\mathcal{B}$ $(\Sigma A, B) \cong \mathcal{B}(A, \Omega B)$ and $[\Sigma A, B] = [A, \Omega B]$.

As a word of caution it should be noted that $\Sigma$ is not the suspen-
From Lemma 3 one deduces

**Proposition 3.** The functor $K^{-n} (n > 0) : \mathcal{B} \to Ab$ is pro-represented by $\Sigma^{n-1}gl$.

We show in addition that $\kappa_{G}^{G}$ is pro-representable in $\mathcal{B}$, where $G$ is a Mayer-Vietoris functor which is an algebraic group. (The definition of $\kappa_{G}^{G}$ in [1] is modified as in Proposition 1 above in the nondiscrete case.)

**References**


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