THE STRUCTURE OF $\omega$-REGULAR SEMIGROUPS

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1. Finding the complete structure of regular semigroups of a certain class has succeeded only when sufficiently strong conditions on idempotents and/or ideals have been imposed. On the one hand, there is the theorem of Rees [7], giving the structure of completely 0-simple semigroups, and its successive generalizations to primitive regular semigroups [2], and 3- and 3i-regular semigroups [4]. On the other hand, with very different restrictions, Reilly [8] has determined the structure of bisimple $\omega$-semigroups, Kočin [1] of inverse simple $\omega$-semigroups, Munn [5] of inverse $\omega$-semigroups.

An $\omega$-chain with zero is a poset \( \{e_i | i \geq 0\} \cup 0 \) with \( e_i > e_j \) if \( i < j \), and \( 0 < e_i \) for all \( i, j \). We call a regular semigroup \( S \) $\omega$-regular if \( S \) has a zero and the poset of its idempotents is an orthogonal sum [2] of $\omega$-chains with zero. We announce here the complete determination of the structure of such semigroups, including various special cases thereof, and briefly mention their isomorphisms.

2. An $\omega$-regular semigroup can be uniquely written as an orthogonal sum of $\omega$-regular prime (i.e., with 0 a prime ideal) semigroups. This reduces the problems of structure and isomorphism to $\omega$-regular prime semigroups. We distinguish three cases: (i) 0-simple, (ii) prime with a proper 0-minimal ideal, (iii) prime without a 0-minimal ideal. Case (i) is the most difficult (and interesting) and includes a variety of special cases some of which reduce to those constructed by Reilly [8], Kočin [1], and Munn [5], [6].

3. Let \( A \) be a nonempty set, \( d \) be a positive integer, \( V \) be a semigroup which is a chain of \( d \) groups \( G_0 > G_1 > \cdots > G_{d-1} \), and \( \sigma \) be a homomorphism of \( V \) into \( G_0 \). Let \( w : A \rightarrow \{0, 1, \ldots, d-1\} \) be any function, denoted by \( w : \alpha \rightarrow w_\alpha \). For \( \alpha \in A \), \( 0 \leq i, j < d \), define \( \langle \alpha, i \rangle \) by

\[
\langle \alpha, i \rangle = w_\alpha + i \pmod{d}, \quad 0 \leq \langle \alpha, i \rangle < d,
\]

and define \([i, \alpha, j]\) to satisfy

\[
[i, \alpha, j]d = (i - j) - (\langle \alpha, i \rangle - \langle \alpha, j \rangle).
\]
Construction I. On the set 
\[ S = \{(\alpha, m, g, n, \beta) | \alpha, \beta \in A, m, n \geq 0, g \in V\} \cup \{0, 1\} \]
define a multiplication by: for \( g_i \in G_i, g_j \in G_j, v = n - s = [i, \beta, j] \),
\[(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m - [i, \alpha, j] - v, (g_i \sigma^{-v})g_j, t, \gamma) \]
if \( v < 0 \), or \( v = 0, i \leq j \);
\[(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m, g_i(g_j \sigma^v), t + [i, \gamma, j] + v, \gamma) \]
if \( v > 0 \), or \( v = 0, i > j \);
and all other products are equal to 0. The set \( S \) with this multiplication will be denoted by \( \Theta(A, \omega; V, \sigma) \).

Construction II. On the set 
\[ S' = \{(\alpha, m, g, n, \beta) | \alpha, \beta \in A, m - w_a = n - w_b = i \pmod d, g \in G_i\} \cup \{0, 1\} \]
define a multiplication by: for \( g_i \in G_i, g_j \in G_j, v = n' - s' = [i, \beta, j] \),
where \( n = n'd + n'', s = s'd + s'', 0 \leq n'', s'' < d \),
\[(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m + s - n, (g_i \sigma^{-v})g_j, t, \gamma) \]
if \( n \leq s \);
\[(\alpha, m, g_i, n, \beta)(\beta, s, g_j, t, \gamma) = (\alpha, m, g_i(g_j \sigma^v), t + n - s, \gamma) \]
if \( n > s \);
and all other products are equal to 0. The set \( S' \) with this multiplication will be denoted by \( \Theta(A, \omega; V, \sigma) \).

The following is our fundamental result.

**Theorem 1.** For a groupoid \( S \), the following statements are equivalent.
(i) \( S \) is a 0-simple \( \omega \)-regular semigroup;
(ii) \( S \) is isomorphic to \( \Theta(A, \omega; V, \sigma) \);
(iii) \( S \) is isomorphic to \( \Theta(A, \omega; V, \sigma) \).

The proof of “(i) \( \Rightarrow \) (ii)” consists of “introducing coordinates” into various \( \mathcal{L} \)- and \( \mathcal{R} \)-classes and of constructing the homomorphism \( \sigma \); it is quite long and is broken into a sequence of lemmas. For “(ii) \( \Rightarrow \) (iii)” one finds a suitable isomorphism, while “(iii) \( \Rightarrow \) (i)” consists of a verification of the defining properties of a 0-simple \( \omega \)-regular semigroup.

Define the \textit{top} of \( S \) in the theorem by
\[ \Theta(S) = \{a \in S | e \mathcal{L} a, a \mathcal{R} f \text{ for some maximal idempotents } e, f\} \cup \{0, 1\} \]
Then \( \Theta(S) \) is a primitive inverse semigroup. It follows from the proof that we can always suppose that \( w_a = 0 \) for some \( \alpha \in A \). Call \( S \) balanced if any two maximal idempotents of \( S \) are \( \mathcal{D} \)-equivalent.

**Theorem 2.** The following conditions on a 0-simple \( \omega \)-regular semigroup \( S \) are equivalent.
(i) \( S \) is balanced;
(ii) \( S \) admits a representation as in Theorem 1 with \( w_a = 0 \) for all \( a \in A \);
(iii) \( \mathcal{S}(S) \) is a Brandt semigroup;
(iv) \( S \) is isomorphic to a Rees matrix semigroup \( \mathcal{M}_0(K; A, A; \Delta) \) over a simple inverse \( \omega \)-semigroup \( K \), \( \Delta \) is the identity matrix.

The structure of the semigroup \( K \) in Theorem 2 was determined by Kočin [1] and Munn [5], the Rees matrix semigroups over bi-simple inverse semigroups were studied in [3] (for the 0-simple case in the theorem, cf. [3, Corollary 5.7] and [6, Theorem 4.2]). Various other special cases include: 0-bisimple, combinatorial, balanced, and combinations thereof.

4. For the remaining cases, we will need the following.

Construction III. Let \( Y \) be a tree semilattice satisfying one of the two conditions: (1) \( Y \) has a zero \( \zeta \) and all elements of \( Y \) are of finite height, (2) \( Y \) has no zero and is of locally finite length. To every non-zero element \( \alpha \) of \( Y \) associate a Brandt semigroup \( S_\alpha \), suppose that the family \( \{S_\alpha\} \) is pairwise disjoint, and that a homomorphism \( \phi_\alpha : S_\alpha \to S_\beta \) is given, where \( \alpha \) is the unique element of \( Y \) covered by \( \alpha \), with the properties:

(i) \( S_\alpha \phi_\alpha \cap S_\beta \phi_\beta = 0 \) if \( \alpha = \beta \);
(ii) for every infinite ascending chain \( \alpha_1 < \alpha_2 < \cdots \) in \( Y \) and every \( a \in S_{\alpha_1}^* \), there exists \( \alpha_0 \) such that \( a \in S_{\alpha_0} \phi_{\alpha_0} \phi_{\alpha_0-1} \cdots \phi_{\alpha_1} \). Let \( \psi_{\alpha, \alpha} \) be the identity mapping on \( S_\alpha \), and for \( \alpha > \beta \), let \( \psi_{\alpha, \beta} = \phi_{\alpha} \phi_{\alpha_1} \cdots \phi_{\alpha_n} \) where \( \alpha > \alpha_1 > \cdots > \alpha_n > \beta \). Let \( S = (U_{a \in Y \setminus \{0\}} (S_a \setminus 0_a)) \cup 0 \) where \( \zeta \) is the zero of \( Y \) (if \( Y \) has one), and 0 is an element not contained in any \( S_a \), and on \( S \) define the multiplication \( * \) by

\[
a * b = (a \psi_{\alpha, \alpha}) (b \psi_{\beta, \alpha}) \text{ if } \alpha \beta \neq \zeta \text{ and } (a \psi_{\alpha, \alpha}) (b \psi_{\beta, \alpha}) \neq 0_{\alpha \beta} \text{ in } S_{\alpha \beta},
\]

and all other products are equal to 0. The set \( S \) with this multiplication will be called a Brandt tree if \( Y \) has a zero and a rooted Brandt tree otherwise.

Theorem 3. A semigroup \( S \) is prime \( \omega \)-regular and has a proper 0-minimal ideal if and only if \( S \) is an ideal extension of a 0-simple \( \omega \)-regular semigroup \( I \) by a Brandt tree \( T \) determined by a 0-restricted homomorphism of \( T \) into the top of \( I \).

Such a homomorphism is completely determined by its restriction to the socle \( \mathcal{S}(T) \) of \( T \), so all such homomorphisms are given by 0-restricted homomorphisms of \( \mathcal{S}(T) \) into \( \mathcal{S}(I) \), both of which are primitive inverse semigroups, and are easy to find explicitly.
Theorem 4. A groupoid $S$ is a prime $\omega$-regular semigroup without 0-minimal ideals if and only if $S$ is a rooted Brandt tree.

5. The semigroups $\mathcal{O}(A, w; V, \sigma)$ and $\mathcal{O}[A, w; V, \sigma]$ do not seem to admit a neat isomorphism theorem except in special cases. In the balanced case, using Theorem 2, [3, 4.1], and [1, Theorem 4], we derive a satisfactory isomorphism theorem. A direct proof does the same in the case these semigroups are combinatorial. Isomorphisms of the semigroups in Construction III are similar to those in [4, Théorème 3.1], while isomorphisms of the semigroups in Theorem 3 can be expressed by isomorphisms of $I$ and $T$ satisfying a commutative diagram.

References


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