I. Introduction. Let $\gamma$ be an oriented, rectifiable Jordan curve in $E^3$ homeomorphic to the unit circle, $u^2 + v^2 = 1$. Let $\Delta$ be the open unit disk, $u^2 + v^2 < 1$, and let $\overline{\Delta}$ be its closure. The classical existence theorem for Plateau's problem as proven by J. Douglas [1], and T. Rado [6] asserts the existence of a minimal surface of the type of the unit disk, whose boundary is $\gamma$, and which has minimum Lebesgue area. The theorem stated in this paper is an extension of this result to surfaces of constant mean curvature.

Let $h(u, v) : \overline{\Delta} \to E^3$ be a given minimal surface solving Plateau's problem. Let $K$ be a given constant and consider the class of continuous vector functions $x : \overline{\Delta} \to E^3$ whose boundary values describe $\gamma$, and such that the oriented volume enclosed by $x$ and $h$ is $K$. We prove that in this class there is an $x$ of minimum Lebesgue area, $x$ is a representation of a surface of constant mean curvature and satisfies the following system of equations.

\begin{align*}
(1) \quad \Delta x &= 2H(x_u \times x_v), \\
(2) \quad |x_u| &= |x_v|, \quad (x_u \cdot x_v) = 0 \quad \text{[conformality]}, \\
(3) \quad x : \partial \Delta &\to E^3 \text{ is an admissible representation of } \gamma.
\end{align*}

Previous existence theorems for the system (1) have been given by E. Heinz [2], H. Werner [8], and S. Hildebrandt [3]. They proved that if $\gamma$ is contained in the unit ball, $x^2 + y^2 + z^2 \leq 1$, and if $H$ with $|H| \leq 1$ is given, then there exists a solution to the system (1) which is itself contained in the unit ball.

We now give a more precise statement of the theorem.

II. Statement of theorem. Denote by $S(\gamma)$ the set of vector functions $x : \overline{\Delta} \to E^3$ continuous on $\overline{\Delta}$, continuously differentiable on $\Delta$, whose boundary values are an admissible representation of the oriented Jordan curve $\gamma$, and such that the "Dirichlet" integral

\begin{equation}
D(x) = \int \int_{\Delta} |x_u|^2 + |x_v|^2 \, du \, dv
\end{equation}


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is finite. We assume that $\mathcal{S}(\gamma)$ is not empty. It is well known that this is true if $\gamma$ is rectifiable, for example.

For each $x \in \mathcal{S}(\gamma)$ the oriented volume functional

$$V(x) = \frac{1}{2} \int_{\Delta} x \cdot (x_u \times x_v) du \, dv$$

is well defined and finite. Also each $x \in \mathcal{S}(\gamma)$ is a representation of a parametric surface whose Lebesgue area does not exceed $D(x)/2$.

**Theorem 1.** Let $K$ be a given constant. Let $\mathcal{S}(\gamma, K)$ denote those members of $\mathcal{S}(\gamma)$ for which $V(x) = K$. There is a member of $\mathcal{S}(\gamma, K)$ of minimum Lebesgue area, which is a representation of a parametric surface of constant mean curvature satisfying the system (1) for some constant $H$.

**III. Indication of proof.** Let $W_1$ be the Sobolev Hilbert space of vector valued functions $x: \Delta \to \mathbb{R}^3$ for which $|x|$, $|x_u|$, and $|x_v|$ are square integrable. As shown by C. B. Morrey [4] each $x \in W_1$ has a well-defined boundary function $x: \partial \Delta \to \mathbb{R}^3$ which is in $L_2(\partial \Delta)$. Let $\mathcal{S}(\gamma)$ denote those members of $W_1$ whose boundary values are an admissible representation of the oriented Jordan curve $\gamma$. From the results in [7] it is known that the oriented volume functional $V(x)$ on $\mathcal{S}(\gamma)$ has a well-defined continuous extension to all of $\mathcal{S}(\gamma)$.

**Theorem 2.** Let $K$ be a given constant. Let $\mathcal{S}(\gamma, K)$ be those members of $\mathcal{S}(\gamma)$ with $V(x) = K$. There is a member of $\mathcal{S}(\gamma, K)$ of minimum "Dirichlet" norm, $D(x)$.

It then follows from the results in [7], that any vector function which solves Theorem 2 also is a solution to our initial theorem.

**Remark.** The results stated here do not preclude the possibility of branch points for our surface (i.e. points where $|x_u| = |x_v| = 0$). Hildebrandt has shown that such points must be isolated in $\Delta$. Recently, R. Osserman [5] has shown that if $h(u, v): \Delta \to \mathbb{R}^3$ is a conformal representation of a minimal surface satisfying the system (1) with $H = 0$ and which minimizes area, then $h$ has no branch points. It would be interesting to know whether or not the same may be said for any vector function which solves Theorem 1.

**Bibliography**


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