SETS OF INTERPOLATION FOR MULTIPLIERS

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Let $T$ denote the circle and $I$ a closed ideal of $L^1(T)$ under convolution. Let $\mathcal{F}I$ denote the set of sequences of complex numbers which are Fourier transforms of elements of $I$.

$$\mathcal{F}I = \{ (\xi_n) : \exists f \in I, \hat{f}(n) = \xi_n \}.$$ 

A subset $E$ of the integers is called a set of interpolation for the multipliers of $\mathcal{F}I$ ($= M(\mathcal{F}I)$) if every bounded complex sequence defined on $E$ is the restriction to $E$ of a multiplier of $\mathcal{F}I$. $E$ is called a Sidon set if every bounded complex sequence on $E$ is the restriction to $E$ of the Fourier transform of some measure on $T$. Answering a question of Y. Meyer we show here that every set of interpolation $E \subseteq \mathbb{Z}^+$ for $M(\mathcal{F}H^1(T))$ is a Sidon set.

Let $A(T)$ denote the Banach space of all analytic continuous functions on $T$ equipped with the supremum norm. Let $\beta = H^1(T) \otimes C(T)$ be the Banach space of all elements of $A(T)$ which can be expressed in the form $\sum_i f_k \ast g_k$ where $f_k \in H^1(T)$, $g_k \in C(T)$ and such that $\sum_i \| f_k \| \| g_k \|_\infty < \infty$. The norm $\| \cdot \|_\beta$ in $\beta$ is the infimum over all such representations. Meyer [1] has shown that the dual of $\beta$ is precisely $M(\mathcal{F}H^1(T))$.

**Theorem 1.** $\beta$ is isometrically isomorphic to $A(T)$.

**Proof.** It is clear that the natural embedding of $\beta$ in $A(T)$ is norm decreasing. Let $P(\theta) = \sum_i a_k \exp[i n_k \theta]$ be an arbitrary analytic trigonometric polynomial and write $e^{i M \theta} P(\theta)$ as

$$\sum_{n=-N}^N \left( 1 - \left| \frac{n}{N} \right| \right) \exp[i (n + N) \theta] \ast \sum_{k=1}^r b_k \exp[i (n_k + M) \theta]$$

where $b_k = a_k \left( 1 - \left| n_k + M - N \right| / N \right)^{-1}$. Choose $M = N - \lfloor N^{1/2} \rfloor$ and $N$ larger than $n$. It is clear that as $N \to \infty$, $b_k \to a_k$ for each $k$. Since the polynomial on the left-hand side is just a translate of the usual Fejer kernel, it has $L^1$ norm equal to 1. By the choice of $M$, the sup norm of the polynomial on the right-hand side tends to $\| P(\theta) \|_\infty$ as $N \to \infty$. Hence

$$\| \exp[i M \theta] P(\theta) \|_\beta < \| P(\theta) \|_\infty + \epsilon$$

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for \( N \) sufficiently large.

It is clear that \(|P(\theta)|_\beta \leq |\exp [iM\theta]P(\theta)|_\beta\) for all positive integers \( M \). Hence \(|P(\theta)|_\beta \leq \|P(\theta)\|_\omega\). Since the analytic trigonometric polynomials are dense in \( \beta \) the theorem follows.

The following answers a question raised in [1, p. 554].

**Corollary.** \( E \subseteq \mathbb{Z}^+ \) is a set of interpolation for \( M(\mathcal{F}H^1(T)) \) if and only if \( E \) is a Sidon set.

**Proof.** The only implication of interest is the "only if" one. Thus assume \( E \) is a set of interpolation for \( M(\mathcal{F}H^1(T)) \). It is an easy consequence of the definition that \( E \) is a set of interpolation for \( M(\mathcal{F}H^1(T)) \) if and only if the elements of \( \beta \) whose spectra are contained in \( E \) have absolutely convergent Fourier series. Hence there is some constant \( c \), depending only on \( E \), such that

\[
\sum_{k=1}^r |a_k| \leq c\|P(\theta)\|_\beta
\]

for all trigonometric polynomials \( P(\theta) = \sum_{k=1}^r a_k \exp [in_k\theta] \) with spectrum contained in \( E \). Since \( \|P(\theta)\|_\omega = \|P(\theta)\|_\beta \), \( E \) is a Sidon set (cf. [3, p. 121]). Q.E.D.

It is of some interest to compare the above notions of interpolation in \( M(\mathcal{F}T) \) with the following definition implicit in [1]: \( E \) is said to be a set of \( E \)-interpolation if every bounded complex sequence on \( E \) is the restriction to \( E \) of a multiplier of \( \mathcal{F}I(E) \) where \( I(E) \) is the ideal of all \( L^1 \) functions whose spectrum is contained in \( E \). The concept of Sidon set is replaced here by that of \( \Lambda(2) \) set. Recall that \( E \) is a \( \Lambda(2) \) set if every \( L^1 \) function whose spectrum is contained in \( E \) is in \( L^2 \).

**Theorem 2.** \( E \) is a set of \( E \)-interpolation if and only if it is a \( \Lambda(2) \) set.

**Proof.** The fact that \( \Lambda(2) \) sets are sets of \( E \)-interpolation is an immediate consequence of the Riesz-Fisher theorem.

Conversely if \( E \) is a set of \( E \)-interpolation and \( P(\theta) = \sum a_k \exp [in_k\theta] \) is an \( E \)-polynomial define \( g(t, \theta) = \sum a_k \varphi_k(t) \exp [in_k\theta] \) where \( \varphi_k \) is the \( k \)th Rademacher function. Then \( g_t = s_t \ast f \) where \( s_t \) is the convolution operator from \( I(E) \) to \( L^1(T) \) such that \( s_t(\pi_k) = \varphi_k(t) \).

Let \( l_\omega, E \) denote the quotient space of \( l_\omega \) by the closed subspace of those sequences vanishing on \( E \). Then since \( E \) is a set of \( E \)-interpolation, the natural map \( \sigma : M(\mathcal{F}I(E)) \to l_\omega, E \) is onto, and hence has a bounded inverse. Thus \( \|s_t\| \leq c \) where \( c \) is independent of \( t \), and \( \|g_t\|_1 \leq c\|f\|_1 \). The proof now proceeds as in Theorem 3.1 of [2].

Integrate \( (\sum |a_k|^2)^{1/2} \leq 2\int_0^\pi |g(t, \theta)| \) with respect to \( \theta \) over \([−\pi, \pi]\) and use the above inequality.
REMARK. In direct analogy to the space $\beta$, $\beta_{\beta} = I(E) \otimes C(T)$ may be formed. It may be of interest to ask for what sets $E$ it is true that whenever $F \subset E$, $F$ is a set of $M(\mathfrak{H}(E))$ interpolation if and only if it is Sidon. By Theorem 2 this will fail if $E = F$ and $E$ is taken to be a set which is $\Lambda(2)$ but not Sidon.

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