QUOTIENTS OF FINITE $W^*$-ALGEBRAS

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1. In this note we present results concerning the following problem. Suppose $M$ is a $W^*$-algebra and $J \subseteq M$ a uniformly closed two-sided ideal. Then the quotient algebra $M/J$ is a $C^*$-algebra, and the problem is: What are the conditions that $M/J$ be a $W^*$-algebra?

Since we can write $M$ as a direct sum of a finite and a properly infinite $W^*$-algebra, we can discuss the two cases separately. In [3] and [4] Takemoto solved the problem for a properly infinite $W^*$-algebra, that can be represented on a separable space. His theorem states that $M/J$ is a $W^*$-algebra, if and only if $J$ is ultra-weakly closed.

2. If $M$ is finite the situation is quite different. There are "many" non-ultra-weakly closed ideals $J$ for which the quotient $M/J$ is a $W^*$-algebra. Indeed, Wright [5] and Feldman [1] proved that if $J$ is a maximal ideal, $M/J$ is a finite factor. This result was proved by a different method by Sakai in [2]. The following theorem generalizes that result.

**Theorem 1.** Let $M$ be a finite and $\sigma$-finite $W^*$-algebra with center $Z$. Let $J$ be a uniformly closed two-sided ideal satisfying the following conditions:

(i) $J$ is an intersection of maximal ideals,
(ii) $Z/Z \cap J$ is a $W^*$-algebra,
(iii) $Z/Z \cap J$ is $\sigma$-finite.

Then $M/J$ is a $W^*$-algebra.

As a partial converse we have

**Theorem 2.** If $J$ is a uniformly closed two-sided ideal of the finite and $\sigma$-finite $W^*$-algebra $M$ and $M/J$ is a $W^*$-algebra, then $J$ satisfies the conditions (i) and (ii) of Theorem 1.

**Remark.** If we assume that $M$ can be represented on a separable
space, and if we further assume the continuum hypothesis, the condition (iii) becomes necessary for $M/J$ to be a $W^*$-algebra. Thus, under these conditions, (i), (ii), and (iii) are necessary and sufficient for $M/J$ to be a $W^*$-algebra.

3. Outline of proofs. The necessity. If $M/J$ is a $W^*$-algebra, it is necessarily a finite $W^*$-algebra. Since the intersection of maximal ideals in the finite $W^*$-algebra $M/J$ is $\{0\}$, $J$ is an intersection of maximal ideals. Moreover, $Z/Z \cap J$ is isomorphic to the center of $M/J$, and is therefore a $W^*$-algebra. Under the continuum hypothesis, a simple cardinality argument shows, that $Z/Z \cap J$ must be $\sigma$-finite, if we also assume that $M$ can be represented on a separable space.

The sufficiency. Suppose that (i), (ii), and (iii) are satisfied. We consider the Banach $Z$-module $\mathcal{E}$ generated by the maps $x \in M \rightarrow (ax)^f \in Z$, where $a \in M$. $\#$ is the canonical center valued trace on $M$. Since $J$ is an intersection of maximal ideals, it is invariant under $\mathcal{E}$, and we can factor $\mathcal{E}$ to $\mathcal{E}$, consisting of linear maps $M/J \rightarrow Z/Z \cap J$. By conditions (ii) and (iii) there is a normal faithful state $\mu$ on $Z/Z \cap J$. Let $F$ be the set of all linear functionals of the form $\mu \circ \Phi$, where $\Phi \in \mathcal{E}$. By Sakai's criterion it suffices to prove that $M/J$ is a dual space of $F$. Let $E$ be the completion of $F$. By a theorem in [2], it suffices to prove that for every $f \in E$, there is an $x \in M/J$ with $\|x\| = 1$ and $f(x) = \|f\|$. By means of polar decomposition for elements of $\mathcal{E}$ this is easily proved, if $f \in F$. In order to extend this result to $E$, we decompose the functionals in $E$ over $Z/Z \cap J$, and by applying a technique similar to that of standard measure theory, we obtain the desired result for $f \in E$.

4. Thus the problem of finding the $W^*$-quotients of the finite $W^*$-algebra $M$ (which is supposed to act on a separable space), is reduced to the corresponding problem for the center $Z$. Since we assume that $M$ acts on a separable space we may assume that $Z = L^{\infty}_m(I)$, the space of essentially bounded Lebesgue measurable functions on the unit interval, or $Z = l^{\infty}(N)$, the set of bounded complex sequences.

Theorem 3. There exist countably many surjective nonnormal $^*$-homomorphisms $\Phi : Z_1 \rightarrow Z_2$ with pairwise different kernels in each of the following three cases:

(i) \( Z_1 = L^{\infty}_m(I), \ Z_2 = L^{\infty}_m(I) \),

(ii) \( Z_1 = L^{\infty}_m(I), \ Z_2 = l^{\infty}(N) \),

(iii) \( Z_1 = l^{\infty}(N), \ Z_2 = l^{\infty}(N) \).

Thus we see that in general there are many non-ultra-weakly closed
ideals $I$ for which $Z/I$ is a $W^*$-algebra. To give nontrivial examples in the noncommutative case we need only consider the finite $W^*$-algebra $F \otimes Z$ where $F$ is a finite factor. The center of $F \otimes Z$ is $Z$, and if $I \subseteq Z$ is an ideal, let $J$ be the ideal of $M$ which is the intersection of the maximal ideals that contain $I$. If $Z/I$ is a $\sigma$-finite $W^*$-algebra, then so is $F \otimes Z/J$, and $J$ is ultra-weakly closed if and only if $I$ is.

5. As to the existence of non $W^*$-quotients we have the following theorem:

**Theorem 4.** Let $Z$ be an infinite dimensional abelian $W^*$-algebra, which can be represented on a separable space. Then $Z$ admits a $C^*$-quotient, which is not a $W^*$-algebra.

**Proof.** If $Z = L^\infty(N)$, $L^\infty(N)/c_0(N)$ will do. If $Z = L_m^\infty(I)$ we apply Theorem 3, part (ii).

**Theorem 5.** (i) There exists a pair $J, M$ such that the center of $M/J$ is a $\sigma$-finite $W^*$-algebra, but $J$ is not an intersection of maximal ideals.

(ii) There exists a pair $J, M$ such that $J$ is an intersection of maximal ideals, but the center of $(M/J)$ is not a $W^*$-algebra.

In the proof of Theorem 5 (ii) we apply Theorem 4. To prove Theorem 5 (i) we need the following considerations. Let $M$ be a finite $W^*$-algebra. The maximal ideal space $\text{Max}(M)$ considered as a subset of the primitive ideal space $\text{Prim}(M)$ is homeomorphic to $\text{Max}(Z)$. Let $x(J)$ be the image of $x \in M$ in $M/J$ under $M \to M/J$, if $J \subseteq \text{Max}(M)$. Then the following is applicable in the proof of Theorem 5 (i).

**Theorem 6.** Let $J_0 \subseteq \text{Max}(M)$. Then the following two conditions are equivalent:

(i) There is no primitive ideal $J$ such that $J \neq J_0$ and $J \cap Z = J_0 \cap Z$.

(ii) The functions $J \subseteq \text{Max}(M) \to \|x(J)\|$, $x \in M$, are continuous at $J_0$.

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Detailed proofs will appear elsewhere.

**References**


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