ON REGULARITY OF GENERALIZED SURFACES
OF CONSTANT MEAN CURVATURE

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Statement of result. A surface in a manifold is called “regular” to emphasize that it is immersed; the term “generalized surface” is used when regularity is only assumed almost everywhere.

We treat generalized surfaces of constant mean curvature in an analytic Riemannian manifold $M$ of dimension 3. To this end consider the variational problem:

$$E[X] = D[X] + 4HV[X] \rightarrow \min$$

where the admissible $X$ are all mappings of a domain $G \subset \mathbb{R}^3$ into $M$, continuous in $G$ and $C^2$ in $G$, which agree with a given admissible $X_0$ on $\partial G$ up to reparameterization; and where $H$ is a real constant, $D[X]$ is the Dirichlet integral of $X$

$$D[X] = \int_0 \int_0 g_{kl}(\dot{x}^k_u \dot{x}^l_u + \dot{x}^k_v \dot{x}^l_v) \, du \, dv$$

and $V[X]$ is volume of a consistently chosen region whose oriented boundary is $X - X_0$. A mapping which is stationary for this functional is a generalized surface of constant mean curvature $H$ in isothermal coordinates, cf. [1, pp. 107–112]. If a mapping is not only stationary but minimizing, we can say more:

**Theorem.** Any mapping $X$ which minimizes $E$ among the admissible mappings defines a regular surface.

Observe that the mapping $X$ itself need not be a regular parameterization. Since the theorem can be applied in any neighborhood, it follows that a mapping which minimizes $E$ even locally must be regular. The theorem is a generalization of the work of Osserman on the Plateau problem, that is, the case $H = 0$, $M = E^3$ [3].

**Consequences.** In the case that $M$ is a space form, examples can be constructed of generalized surfaces of constant mean curvature with


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singularities resembling branch points of conformal mappings. This construction was given by Courant for the case $H=0, M=E^3$ [1, pp. 122ff]. By the above theorem these generalized surfaces are unstable with respect to $E$ in every neighborhood of the singular point, in contrast to what is known for single-integral variational problems.

Combining the above theorem with various results on the existence of generalized surfaces as solutions to variational problems of this type improves each to a statement of existence for regular surfaces. For example, the result of Hildebrandt [2] now yields

**Theorem.** Given a continuous curve $\gamma$ in the unit ball of $E^3$ such that some mapping with finite Dirichlet integral spans $\gamma$, and given $H$ with $|H| \leq 1$, there exists a regular surface spanning $\gamma$ with constant mean curvature $H$.

This answers a question posed in [2], and, in similar fashion, the conjecture in [3, §5.4] for the analytic case.

**Indication of proof.** The approach taken is to study the behavior of a solution to the Euler equations near a branch point, i.e. a point of nonregularity. Orthogonal projection on a certain plane is shown to be a covering map in a punctured neighborhood. Properties of nonparametric solutions are then applied toward an analysis of the self-intersections of the solution. We prove

**Theorem.** There is an analytic arc of self-intersection of the solution which converges to the branch point in a well-defined direction.

**Lemma.** There is a neighborhood of the branch point in which either

1. the solution intersects itself only transversally; or
2. with appropriate choice of parameters, the solution is invariant under a rotation of the parameter domain.

The main result is then obtained in case (1) by means of a discontinuous "reparameterization" near the branch point which leaves the value of $E$ unchanged but introduces a corner. This new generalized surface cannot minimize $E$, nor, therefore, can the original solution. In case (2) the solution has an inessential branch point; the surface itself is regular.

This work can be generalized to a broad class of geometrically meaningful variational problems whose Euler equations form elliptic systems. In particular, geometric side conditions may be imposed.

Details will appear in the author's thesis.
BIBLIOGRAPHY


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