FUNCTION ALGEBRAS AND THE DE RHAM
THEOREM IN PL

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0. Introduction. There is a classical contravariant functor on the
category of smooth manifolds \( M \) which assigns to each \( M \) the algebra
\( A \) of all smooth functions on \( M \), and one uses this functor implicitly
throughout differential topology. For example, the de Rham theorem
extends the customary derivation \( d: A \rightarrow \mathcal{C}(A) \) to a cochain complex
\( (\Delta \mathcal{C}(A), d) \) whose homology is isomorphic to the real cohomology of
\( M \) itself. In this paper we construct a corresponding contravariant
functor on the category of piecewise linear manifolds \( M \), which as­
signs to each \( M \) an algebra \( A \) of functions on \( M \). We then define a
derivation \( d: A \rightarrow \mathcal{C}(A) \) and extend it to a cochain complex \( (\Delta \mathcal{C}(A), d) \)
whose homology is isomorphic to the real cohomology of \( M \); this is
the de Rham theorem in PL. As an application we construct connec­
tions and curvature homomorphisms in terms of \( (\Delta \mathcal{C}(A), d) \), to which
we apply a real version of the Chern-Weil theorem to compute real
Pontrjagin classes of PL manifolds without using the Hirzebruch
L-polynomials.

1. Smoothing homeomorphisms. A simplicial decomposition of
\( \mathbb{R}^n \) at 0 is any finite triangulation of \( \mathbb{R}^n \) into open simplexes such that
\( 0 \in \mathbb{R}^n \) is the only 0-simplex. If \( \alpha \) and \( \beta \) are any two such simplicial
decompositions then we write \( \alpha < \beta \) whenever \( \beta \) is a subdivision of \( \alpha \).
For any \( \alpha \) and \( \beta \) there is a simplicial decomposition \( \gamma \) with \( \alpha < \gamma \) and
\( \beta < \gamma \), so that the simplicial decompositions of \( \mathbb{R}^n \) at 0 form a directed
set.

It is clear that a simplicial decomposition \( \alpha \) is completely deter­
mined by its 1-simplexes \( \rho_1, \ldots, \rho_N \) (for some \( N > n \)), each \( p \)-simplex
of \( \alpha \) containing precisely \( p \) 1-simplexes \( \rho_{i_1}, \ldots, \rho_{i_p} \) in its closure. If
\( \mathbb{R}^n \) is endowed with its usual euclidean norm then points on each
1-simplex \( \rho_i \) can be identified with their norms \( x_i \in \mathbb{R}^+ \), and points in
the open \( p \)-simplex determined by \( \rho_{i_1}, \ldots, \rho_{i_p} \) can be identified by the
coordinates \( (x_{i_1}, \ldots, x_{i_p}) \in (\mathbb{R}^+)^p \).

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Now define \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \) by setting \( \varphi(x) = \exp \left( \frac{1}{2} (x - x^{-1}) \right) \), and observe that \( \varphi \) is a diffeomorphism with inverse \( \varphi^{-1} \) given by \( \varphi^{-1}(y) = \ln y + (1 + \ln^2 y)^{1/2} \). One can extend \( \varphi \) to a homeomorphism \( \overline{\varphi} : \mathbb{R}^+ \to \mathbb{R}^+ \) by setting \( \overline{\varphi}(0) = 0 \). Then there is a diffeomorphism \( \Phi \) of each open \( \rho \)-simplex of \( \alpha \) into itself given by \( \Phi(x_{i_1}, \ldots, x_{i_p}) = (\varphi(x_{i_1}), \ldots, \varphi(x_{i_p})) \), and a homeomorphism of the closure into itself given by \( \Phi^{-1}(x_{i_1}, \ldots, x_{i_p}) = (\varphi(x_{i_1}), \ldots, \varphi(x_{i_p})) \). Since \( \overline{\varphi}(0) = 0 \) it follows that the homeomorphisms \( \Phi \) agree on intersections of closures of simplexes in \( \alpha \), hence that the diffeomorphisms \( \Phi \) in all dimensions induce a homeomorphism \( \mathbb{R}^n \to \mathbb{R}^n \).

In the following definition we replace the latter homeomorphism by its \( N \)-fold composition, where \( N \) is the number of 1-simplexes in \( \alpha \), and for convenience we let \( \varphi_N \) denote the \( N \)-fold composition of \( \varphi : \mathbb{R}^+ \to \mathbb{R}^+ \).

**Definition.** For the simplicial decomposition \( \alpha \) of \( \mathbb{R}^n \) at 0, with 1-simplexes \( \rho_1, \ldots, \rho_N \), the smoothing homeomorphism \( \Phi_\alpha : \mathbb{R}^n \to \mathbb{R}^n \) is given by setting \( \Phi_\alpha(x) = 0 \) and \( \Phi_\alpha(x_{i_1}, \ldots, x_{i_p}) = (\varphi_N(x_{i_1}), \ldots, \varphi_N(x_{i_p})) \) for points \( (x_{i_1}, \ldots, x_{i_p}) \) of open \( \rho \)-simplexes in \( \alpha \) with the 1-simplexes \( \rho_{i_1}, \ldots, \rho_{i_p} \) on their boundaries.

2. Smoothable function algebras. In this section we assign an algebra \( A \) of continuous functions \( f : M \to \mathbb{R}^1 \) to each PL manifold \( M \). We begin by taking \( M = \mathbb{R}^n \), for which we first define the algebra \( A_0 \) of germs of elements of \( A \) at \( 0 \in \mathbb{R}^n \).

**Lemma.** If \( f : \mathbb{R}^n \to \mathbb{R}^1 \) is a piecewise linear function which is linear on each simplex of a simplicial decomposition \( \alpha \) of \( \mathbb{R}^n \) at 0, then \( f \circ \Phi_\alpha : \mathbb{R}^n \to \mathbb{R}^1 \) is everywhere smooth.

The proof of the Lemma depends primarily on the following observation: if \( f : \mathbb{R}^1 \to \mathbb{R}^1 \) is any function which is smooth except at 0 \( \in \mathbb{R}^1 \), and if the derivatives of \( f \) are bounded in a deleted neighborhood of 0, then \( \lim_{x \to 0} (f \circ \varphi)(x) = 0 \) for each \( q > 0 \). The details will appear in [6].

The conclusion of the Lemma holds for a very broad class of functions on \( \mathbb{R}^n \), including many functions \( f : \mathbb{R}^n \to \mathbb{R}^1 \) for which the derivatives of \( f \circ \Phi_\alpha \) satisfy no conditions other than smoothness. We let \( \Phi_\alpha^{-1}(C^\infty(\mathbb{R}^n)) \) represent the algebra of those \( f : \mathbb{R}^n \to \mathbb{R}^1 \) such that \( f \circ \Phi_\alpha \in C^\infty(\mathbb{R}^n) \). It will be shown in [6] that if \( \beta > \alpha \) then the homeomorphism \( \Phi_\alpha^{-1}(\Phi_\beta) : \mathbb{R}^n \to \mathbb{R}^n \) is smooth, although its jacobian vanishes on some subset of \( \mathbb{R}^n \); a fortiori \( \Phi_\alpha^{-1}(C^\infty(\mathbb{R}^n)) \subseteq \Phi_\beta^{-1}(C^\infty(\mathbb{R}^n)) \).

**Definition.** A continuous function \( f : \mathbb{R}^n \to \mathbb{R}^1 \) is smoothable at 0 \( \in \mathbb{R}^n \) if its germ at 0 is the germ of an element of \( \Phi_\alpha^{-1}(C^\infty(\mathbb{R}^n)) \) for some simplicial decomposition \( \alpha \) of \( \mathbb{R}^n \). A continuous function \( f : \mathbb{R}^n \to \mathbb{R}^1 \) is smoothable at \( P \in \mathbb{R}^n \) if \( f \circ \tau \) is smoothable at 0 for the
translation $\tau: \mathbb{R}^n \rightarrow \mathbb{R}^n$ carrying $0$ into $P$. For any open subset $V \subset \mathbb{R}^n$, a continuous function $f: V \rightarrow \mathbb{R}^1$ is smoothable on $V$ if and only if it is smoothable at each $P \in V$.

Trivially, if $f: V \rightarrow \mathbb{R}^1$ is smooth then it is also smoothable, and the Lemma implies that if $f: V \rightarrow \mathbb{R}^1$ is piecewise linear then it is also smoothable.

The smoothable functions on any open $V \subset \mathbb{R}^n$ form an algebra. For if $f$ and $g$ are smoothable at $0$ then the germs of $f \circ \Phi_a$ and $g \circ \Phi_b$ are smooth at $0$ for some $\alpha$ and $\beta$, and so are the germs of $f \circ \Phi_{\gamma}$ and $g \circ \Phi_{\gamma}$ for any $\gamma > \alpha$ and $\gamma > \beta$, so that $(f-g) \circ \Phi_\gamma$ and $f \cdot g \circ \Phi_\gamma$ are smooth. In fact, the algebra $A_0$ of germs of smoothable functions at $0 \in \mathbb{R}^n$ is precisely the direct limit $\lim_{\alpha} \Phi_\alpha^{-1}(C^\infty(\mathbb{R}^n))_0$, where the subscript 0 indicates germs at 0, and where each $\Phi_\alpha^{-1}(C^\infty(\mathbb{R}^n))_0 \rightarrow \Phi_\beta^{-1}(C^\infty(\mathbb{R}^n))_0$ is an inclusion homomorphism for $\alpha < \beta$.

We recall that one can define PL manifolds in terms of atlases and PL homeomorphisms of open sets in $\mathbb{R}^n$ just as one defines smooth manifolds in terms of atlases and diffeomorphisms of open sets in $\mathbb{R}^n$. (See [8], for example.) Specifically, one covers a PL manifold $M$ with open sets $U_i$ for which there are homeomorphisms $\Psi_i: U_i \rightarrow V_i$ onto open sets $V_i \subset \mathbb{R}^n$, and the compositions $(\Psi_j|U_i \cap U_j) \circ (\Psi_i|U_i \cap U_j)^{-1}$ are required to be PL homeomorphisms. It will be shown in [6] that the composition of a PL homeomorphism with a smoothable function is smoothable, so that the following definition makes sense:

**Definition.** For any PL manifold $M$ let $\{U_i\}$ be an atlas with homeomorphisms $\Psi_i: U_i \rightarrow V_i$ onto open sets $V_i \subset \mathbb{R}^n$ as above. Then the smoothable function algebra of $M$ consists of those continuous $f: M \rightarrow \mathbb{R}^1$ such that $(f|U_i) \circ \Psi_i^{-1}$ is smoothable on $V_i$ for each $i$.

3. Differential forms in PL. The usual derivation $d: C^\infty(\mathbb{R}^n) \rightarrow \mathcal{E}(C^\infty(\mathbb{R}^n))$ is given by

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n,$$

and it can be described in terms of corresponding derivations of the algebras $C^\infty(\mathbb{R}^n)_P$ of germs of $C^\infty(\mathbb{R}^n)$ at each $P \in \mathbb{R}^n$. This is a general phenomenon about derivations of arbitrary function algebras $A$: any derivation $d: A \rightarrow E$ may be regarded as a section of a sheaf of derivations of the sheaf $A$ of germs $A_P$ of $A$ at points $P$ of the maximal spectrum of $A$ into a sheaf of modules over $A$. (See [5], for example.) We consider only sections over the subset $M$ of the maximal spectrum when $A$ is the smoothable function algebra of $M$.

Any homomorphism $C^\infty(\mathbb{R}^n)_0 \rightarrow C^\infty(\mathbb{R}^n)_0$ of the germs of smooth
functions gives rise to a $C^\infty(\mathbb{R}^n)_\alpha$-module homomorphism $\varepsilon(C^\infty(\mathbb{R}^n)_\alpha) \to \varepsilon(C^\infty(\mathbb{R}^n)_\alpha)$ which commutes with $d$. In particular if $\alpha$ and $\beta$ are simplicial decompositions of $\mathbb{R}^n$ at 0 with $\alpha < \beta$ then $\Phi^\alpha_\beta : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map which induces a homomorphism $C^\infty(\mathbb{R}^n)_\alpha \to C^\infty(\mathbb{R}^n)_\beta$. For convenience we rewrite the resulting commutative diagram

$$
\begin{array}{ccc}
C^\infty(\mathbb{R}^n)_\alpha & \to & C^\infty(\mathbb{R}^n)_\beta \\
d \downarrow & & \downarrow d \\
\varepsilon(C^\infty(\mathbb{R}^n)_\alpha) & \to & \varepsilon(C^\infty(\mathbb{R}^n)_\beta) \\
\varepsilon_\alpha(A_0) & \to & \varepsilon_\beta(A_0)
\end{array}
$$

in the form

$$
\lim_\alpha d_\alpha : A_0 \to \varepsilon(A_0)
$$

to obtain a derivation $d_\alpha : A_0 \to \varepsilon(A_0)$ of the form

$$
\lim_\alpha \Phi^\alpha_\beta(C^\infty(\mathbb{R}^n)_\alpha) \to \lim_\alpha \varepsilon(A_0).
$$

Replacing $0 \in \mathbb{R}^n$ by arbitrary $P \in \mathbb{R}^n$ we then induce a derivation $d : A \to \varepsilon(A)$ of the smoothable function algebra $A$ on $\mathbb{R}^n$. The method of §2 then provides a derivation $d : A \to \varepsilon(A)$ of the smoothable function algebra $A$ on any PL manifold $M$.

4. The de Rham theorem in PL. Since exterior products commute with direct limit, and since the (acyclic) cochain complex $(\wedge \varepsilon(C^\infty(\mathbb{R}^n)_\alpha), d)$ is classically defined for the algebra $C^\infty(\mathbb{R}^n)_\alpha$ it follows that one can form the exterior algebra $\Lambda \varepsilon(A)$ and the cochain complex $(\Lambda \varepsilon(A), d)$ for any algebra $A$ of smoothable functions.

**Theorem.** If $A$ is the smoothable function algebra on a PL manifold $M$, then the homology of $(\Lambda \varepsilon(A), d)$ is isomorphic to the real cohomology of $M$.

**Proof.** It will suffice to establish the PL analog of the Poincaré lemma: the usual sheaf-theoretic argument then applies to the PL case as well as the smooth case. (See [3], for example, which quotes the Poincaré lemma but otherwise invokes no properties of smooth function algebras not also shared by smoothable function algebras; smoothable partitions of unity present no problem.) However, the classical Poincaré lemma states that $(\Lambda \varepsilon(C^\infty(\mathbb{R}^n)_\alpha), d)$ is acyclic, and we shall use this result to obtain the analogous result that $(\Lambda \varepsilon(A_0), d)$ is acyclic for the algebra $A_0 = \lim_\alpha \Phi^\alpha_\beta(C^\infty(\mathbb{R}^n)_\alpha)$ of germs of smoothable functions at $0 \in \mathbb{R}^n$. If

$$
\theta = \sum_{a} f_a^1 \, df_1^a \wedge \cdots \wedge df_p^a \in \Lambda^p \varepsilon(A_0)
$$

for $f_j^a \in A_0
$

then since $A_0$ is a direct limit there is a simplicial decomposition $\alpha$ of
$R^n$ at 0 for which each $f_j \circ \Phi_\alpha$ is smooth. If in addition $d\theta = 0$ then $d(\theta \Phi_\alpha) = 0$; that is,

$$d(\theta \Phi_\alpha) = d \sum_q g_q^j \, dg_1^q \wedge \cdots \wedge d \delta_F = 0 \in \Lambda^{p+1} \mathcal{E}(C^\infty(R^n)_0),$$

where $g_j^i = f_j^i \circ \Phi_\alpha$. The classical Poincaré lemma then provides

$$\psi = \sum_h^0 \, dh_1 \wedge \cdots \wedge dh_{p-1} \in \Lambda^{p-1} \mathcal{E}(C^\infty(R^n)_0) \quad \text{with} \quad d\psi = \theta \Phi_\alpha,$$

where $h_j^i \in C^\infty(R^n)_0$. It follows for $h_j^i = h_j^i \circ \Phi_\alpha^{-1} \in A_\alpha$ and $\psi_\alpha^{-1} = \sum_h^0 \, dh_1 \wedge \cdots \wedge dh_{p-1} \in \Lambda^{p-1} \mathcal{E}(A)$ that $d(\psi_\alpha^{-1}) = \theta$. This completes the Poincaré lemma in PL, hence the de Rham theorem in PL.

5. Pontrjagin classes in PL. Let $A$ be the smoothable function algebra of a PL manifold $M$, and let $\mathcal{F}$ be a graded left $\Lambda \mathcal{E}(A)$-module. Then $\mathcal{F}$ is also a right $\Lambda \mathcal{E}(A)$-module with respect to the product $\Phi\theta = (-1)^{p+1} \theta \Phi$ for $\Phi \in \mathcal{F}^{(p)}$ and $\theta \in \Lambda \mathcal{E}(A)$, and one can form the tensor algebra $\otimes_{\Lambda \mathcal{E}(A)} \mathcal{F}$. Let $\mathcal{I} \subseteq \otimes_{\Lambda \mathcal{E}(A)} \mathcal{F}$ be the two-sided ideal generated by elements $\Phi \otimes \Psi + (-1)^{(p+1)(q+1)} \Psi \otimes \Phi$ for $\Phi \in \mathcal{F}^{(p)}$ and $\Psi \in \mathcal{F}^{(q)}$, and let $\Lambda_{\Lambda \mathcal{E}(A)} \mathcal{F}$ be the quotient of $\otimes_{\Lambda \mathcal{E}(A)} \mathcal{F} / \mathcal{I}$. For example, if $\mathcal{F} = \Lambda \mathcal{E}(A) \otimes \mathcal{E}(A)$ then $\Lambda_{\Lambda \mathcal{E}(A)} \mathcal{F} = \Lambda \mathcal{E}(A) \otimes \Lambda \mathcal{E}(A)$. If $\mathcal{F}$ is locally a direct limit of free $\Lambda \mathcal{E}(C^\infty(R^n)_0)$-modules as in the preceding example, then for any two-sided $\Lambda \mathcal{E}(A)$-module homomorphism $K : \mathcal{F} \to \mathcal{F}$ one can define $\det K \in \Lambda \mathcal{E}(A)$; in this case $\mathcal{F}$ admits determinants.

A connection in a left $\Lambda \mathcal{E}(A)$-module $\mathcal{F}$ is any real linear map $D : \mathcal{F} \to \mathcal{F}$ of degree +1 such that $D \Phi \mathcal{F} = d \Phi + (-1)^p \theta \cdot D \Phi$ for $\theta \in \Lambda^p \mathcal{E}(A)$. The curvature $K$ of $D$ is the composition $D \circ D$, trivially $\Lambda \mathcal{E}(A)$-linear on each side. Here is a very general real Chern-Weil theorem, whose proof will appear in [6]:

**Proposition.** If $\mathcal{F}$ admits determinants then $\det(I + K/2\pi)$ is closed for any connection $D$, and the de Rham cohomology class $[\det(I + K/2\pi)]$ is an element of $H^{4*}(M)$ which is independent of $D$.

We have already observed that $\Lambda \mathcal{E}(A) \otimes \mathcal{E}(A)$ admits determinants.

**Lemma.** $\Lambda \mathcal{E}(A) \otimes \mathcal{E}(A)$ has at least one connection.

Now recall from [4] or [7] that the total rational (or real) Pontrjagin class $\rho(M)$ of a PL manifold $M$ is constructed via the Hirzebruch $L$-polynomials in such a way that if $M$ happens to carry a smooth structure then $\rho(M)$ is the Pontrjagin class of the tangent bundle $\tau(M)$. The following construction avoids the $L$-polynomials; its proof will appear in [6].
Theorem. Let $A$ be the smoothable function algebra of a PL manifold $M$ and let $\mathcal{S}$ be the left $\Delta^*(A)$-module $\Delta^*(A) \otimes \mathcal{S}(A)$; then $[\det(I+K/2\pi)] \in H^{*+}(M)$ is the Pontrjagin class of $M$.

We remark that one can probably establish a PL version of the Gauss-Bonnet theorem within the framework of the present paper, which would more closely parallel the classical formula of [2] than the polyhedral results of [1].

References


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