A NOTE ON COBORDISM OF POINCARE DUALITY SPACES

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1. Introduction. Let $\Omega_n^{PD}$ denote the group of cobordism classes of oriented Poincaré duality spaces of dimension $n$. (See [2] for definitions.) The Pontrjagin-Thom construction yields a natural homomorphism $p: \Omega_n^{PD} \to \pi_n(MSG)$ where $MSG$ is the Thom spectrum associated to the universal spherical fibration over $BSG$.

N. Levitt [2] has shown that if $n \equiv 3 \pmod{4}$, then $p$ is surjective, and if $n \equiv 3 \pmod{4}$, then $\text{cokernel}(p) \subseteq \mathbb{Z}_2$. More precisely, Levitt has shown that, if $n \geq 3$, there is a subgroup $\tilde{\Omega}_n \subseteq \Omega_n^{PD}$ (it is likely that $\Omega_n = \Omega_n^{PD}$) and an exact sequence

$$
\cdots \to P_n \to \tilde{\Omega}_n \xrightarrow{p} \pi_n(MSG) \to P_{n-1} \to \cdots
$$

where $P_n = \mathbb{Z}, 0, \mathbb{Z}_2, 0$ as $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively. Further, image$(P_n) \subseteq \Omega_n^{PD}$ is generated by the cobordism class $[K^n]$ where, if $n \equiv 0 \pmod{4}$, $K^n$ is the almost parallelizable Milnor manifold of index 8, and, if $n \equiv 2 \pmod{4}$, $K^n$ is the almost parallelizable Kervaire manifold constructed by plumbing together the tangent bundles of two $(n/2)$-spheres. ($K^4$ is not a manifold, but it is a Poincaré duality space.)

Our main results, proved in §2, are the following.

**Theorem 1.1.** The Kervaire manifold, $K^{4k+2}$, bounds a Poincaré duality space.

**Theorem 1.2.** The Milnor manifold, $K^{4k}$, is Poincaré duality cobordant to $8(CP(2))^k$.

It follows from Theorem 1.1 that the long exact sequence (1.0) contains short exact sequences

$$
0 \to \tilde{\Omega}_{4k+2} \to \pi_{4k+2}(MSG) \to \mathbb{Z}_2 \to 0.
$$

Our proof of Theorem 1.1 can be formulated to show that this sequence is actually split exact.

Theorem 1.2 describes the short exact sequences

$$
0 \to \mathbb{Z} \to \tilde{\Omega}_{4k} \to \pi_{4k}(MSG) \to 0
$$


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which occur in (1.0). For, $\mathfrak{N}_4$ is a direct sum of $\mathbb{Z}$ and the subgroup of $\mathfrak{N}_4$ of elements of index zero and $[(CP(2))^k]$ can be chosen as a generator of the summand $\mathbb{Z}$.

Since it is not known if $\mathfrak{N}_n = \Omega^\text{PD}_n$, it does not follow immediately that the cokernel of $p: \Omega^\text{PD}_{4k+3} \rightarrow \pi_{4k+3}(MSG)$ is $\mathbb{Z}_2$. However, this is, in fact, the case and in §3 we outline a second proof of Theorem 1.1 (actually, the original proof), due to the first-named author of this note, which shows this additional fact.

2. Proof of Theorems 1.1 and 1.2. Suppose given a diagram

\[
\begin{array}{ccc}
\nu_N & \xrightarrow{f} & \xi_M \\
\downarrow & & \downarrow \\
N^n & \xrightarrow{f} & M^n
\end{array}
\]

where $N^n$ and $M^n$ are closed, oriented manifolds, $\nu_N$ is the normal bundle of $N^n$, $\xi_M$ is a bundle fibre homotopy equivalent to the normal bundle of $M^n$, $f$ is a map of degree one, and $\hat{f}$ is a bundle map covering $f$. Then there is associated a surgery obstruction, $s(N^n, \hat{f}) \in P_n$, to constructing a homotopy equivalence cobordant to $(N^n, \hat{f})$. The surgery obstruction, $s$, satisfies the following product formula of Sullivan [3]:

\[s(L^k \times N^n, 1 \times \hat{f}) = \text{index}(L^k) \cdot s(N^n, f).\]

If $n \equiv 0 \pmod{4}$, then $s(N^n, \hat{f}) = (1/8)(\text{index}(N^n) - \text{index}(M^n))$.

Now, Theorem 1.1 is obvious if $k = 0$ since $K^2 = S^1 \times S^1 = T^2$. Also, there is a well-known normal map

\[
\begin{array}{ccc}
\nu_{T^2} & \xrightarrow{f} & \nu_S^2 \\
\downarrow & & \downarrow \\
T^2 & \xrightarrow{f} & S^2
\end{array}
\]

with $s(T^2, \hat{f}) = 1$. Then $s(CP(2k) \times T^2, 1 \times \hat{f}) = 1$, and the technique of surgery can be used to construct (a normal) cobordism from $1 \times f: CP(2k) \times T^2 \rightarrow CP(2k) \times S^2$ to $g: W^{4k+2} \rightarrow CP(2k) \times S^2$ where $W^{4k+2}$ is the connected sum of $K^{4k+2}$ with a PL manifold $V^{4k+2}$ homotopy equivalent to $CP(2k) \times S^2$. Clearly, $W^{4k+2}$ is a smooth boundary since it is cobordant to $CP(2k) \times T^2$. But $V^{4k+2}$ bounds a Poincaré duality space because it is homotopy equivalent to $CP(2k) \times S^2$. Thus, the difference $[W^{4k+2}] - [V^{4k+2}] = [K^{4k+2}] = 0$. This proves Theorem 1.1.

For Theorem 1.2, we distinguish the cases $k = 1$ and $k > 1$. If $k = 1$,
this has been shown by Wall, and follows from the fact that the index homomorphism $\Omega^\text{PD}_k \to \mathbb{Z}$ is an isomorphism. If $k > 1$, we proceed as follows. Let $H$ denote the canonical complex line bundle over $CP(2)$. Then $24H$ is fibre homotopically trivial. Hence, there is a manifold $N^4$ and a diagram

\[
\begin{array}{ccc}
N^4 & \xrightarrow{f} & CP(2) \\
\downarrow & & \downarrow \\
N^4 & \xrightarrow{\xi} & CP(2)
\end{array}
\]

where $\xi = \nu_{CP(2)} - 24H$. By the Hirzebruch index theorem, $\text{index}(N^4) = 9$, and hence $s(N^4, f) = (1/8)(\text{index}(N^4) - \text{index}(CP(2))) = 1$. Also, $N^4$ is smoothly cobordant to $9(CP(2))$. By the product formula, $s((CP(2))^{k-1} \times N^4, 1 \times f) = 1$. Again by surgery, $1 \times f : (CP(2))^{k-1} \times N^4 \to (CP(2))^{k-1} \times CP(2) = (CP(2))^k$ is cobordant to $g : W^{4k} \to (CP(2))^k$ where $W^{4k}$ is the connected sum of $K^{4k}$ and a PL manifold $V^{4k}$ homotopy equivalent to $(CP(2))^k$. Since $W^{4k}$ is smoothly cobordant to $(CP(2))^{k-1} \times N^4$, hence to $9(CP(2))^k$, and since $V^{4k}$ is Poincaré duality cobordant to $(CP(2))^k$, it follows that the difference $[V^{4k}] - [W^{4k}] = [K^{4k}]$ is Poincaré duality cobordant to $9(CP(2))^k$. This proves Theorem 1.2.

3. Additional comments. Let $K(Z_2, 2k+1) \to BSG(<v_{2(k+1)}>) \to BSG$ be the fibration which kills the Wu class $v_{2(k+1)} \in H^{2(k+1)}(BSG, Z_2) [1]$. Let $MSG(<v_{2(k+1)}>)$ be the Thom spectrum associated to the universal bundle pulled back to $BSG(<v_{2(k+1)}>)$. If $M^{4k+3}$ is a Poincaré duality space, then $v_{2(k+1)}(M^{4k+3}) = 0$; hence the classifying map for the normal spherical fibration, $M^{4k+3} \to BSG$, lifts to a map $M^{4k+3} \to BSG(<v_{2(k+1)}>)$. It follows that if the Pontrjagin-Thom homomorphism $p : \Omega^\text{PD}_{4k+3} \to \pi_{4k+3}(MSG)$ is surjective, then the natural homomorphism $\pi_{4k+3}(MSG(<v_{2(k+1)}>)) \to \pi_{4k+3}(MSG)$ is also surjective.

In [1] it is shown that there is an exact sequence

\[
0 \to Z_2 \xrightarrow{i} \pi_{4k+3}(MSG, MSG(<v_{2(k+1)}>)) \xrightarrow{f} H_{2k+1}(MSG, Z_2) \to 0.
\]

It can further be shown that

\[
\text{image}(\pi_{4k+3}(MSG) \to \pi_{4k+3}(MSG, MSG(<v_{2(k+1)}>))) = i(Z_2) = Z_2.
\]

In particular, $\pi_{4k+3}(MSG(<v_{2(k+1)}>)) \to \pi_{4k+3}(MSG)$ is not surjective; hence $p : \Omega^\text{PD}_{4k+3} \to \pi_{4k+3}(MSG)$ is not surjective.

This argument provides a homotopy theoretic description of Levitt’s obstruction to transversality, $\pi_{4k+3}(MSG) \to Z_2$, which occurs in the exact sequence (1.0). Namely, with the identification of their
images, Levitt's homomorphism coincides with the homomorphism
\[ \pi_{4k+3}(MSG) \rightarrow \pi_{4k+3}(MSG, MSG(v_{2k+1})). \]

REFERENCES


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