C*-ALGEBRAS GENERATED BY MEASURES

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We announce here some results dealing with nonabelian extensions of the theory of almost periodic functions to the duals of compact groups. For $G$ a locally compact group, let $\hat{G}$ be the dual of $G$ (the set of equivalence classes of continuous, irreducible, unitary representations of $G$). For $\pi \in \hat{G}$ and $\mu \in M(G)$, the measure algebra of $G$, let $\pi(\mu)$ be the Fourier-Stieltjes transform of $\mu$ at $\pi$. Let $\|\mu\|_\infty$ be $\sup\{\|\pi(\mu)\| : \pi \in \hat{G}\}$, and let $\mathfrak{M}(\hat{G})$ be the C*-completion of $M(G)$ relative to the norm $\|\cdot\|_\infty$. Let $\mathfrak{M}_a(\hat{G})$, $\mathfrak{M}_d(\hat{G})$ be the closures in $\mathfrak{M}(\hat{G})$ of $L^1(G)$ (the space of measures absolutely continuous with respect to left Haar measure), $M_d(G)$ (the space of discrete measures) respectively. The algebra $\mathfrak{M}_d(\hat{G})$ is a nonabelian analogue of the classical algebra of almost periodic functions. A standard reference for C*-algebras is [1].

We denote the spectrum of $\mathfrak{M}(\hat{G})$ by $\kappa \hat{G}$. In the abelian case this is the closure of the dual group of $G$ in the spectrum of $M(G)$. In general $\hat{G}$ is identified with a dense open subset of $\kappa \hat{G}$ and $\kappa \hat{G}\setminus \hat{G}$ is the annihilator of $\mathfrak{M}_a(\hat{G})$. We investigate the C*-extension of the canonical projection which maps a measure to its discrete part. This makes possible a proof that $\kappa \hat{G}\setminus \hat{G}$ contains a homeomorphic copy of the reduced dual of $G_d$, the group $G$ made discrete. We further show that if $G$ is nondiscrete and $G_d$ is amenable then the sup and lim sup norms are identical on $\mathfrak{M}_d(\hat{G})$, and if $\mu \in \mathfrak{M}_d(\hat{G})$ then $\mu \in M_d(\hat{G})$ ($\mu \in M(G)$).

For $S \subset \kappa \hat{G}$ let $\mathfrak{M}(S) = \{\phi \in \mathfrak{M}(\hat{G}) : \pi(\phi) = 0 \text{ for all } \pi \in S\}$. Then $\mathfrak{M}(S)$ is a closed ideal in $\mathfrak{M}(\hat{G})$. Let $\mathfrak{M}(\hat{G}) = \mathfrak{M}(\hat{G})/\mathfrak{M}(S)$ be the quotient C*-algebra.

Denote the locally compact group $G$ made discrete by $G_d$. Then $G_d$ is the dual of $G_d$ and is also the spectrum of $\mathfrak{M}(G_d) = \mathfrak{M}_d(\hat{G})$. Each $\pi \in \hat{G}$ gives an irreducible unitary representation of $G_d$; thus $\hat{G}$ is identified with a subset of $\hat{G}_d$. We denote the closure of $\hat{G}$ in $\hat{G}_d$ by $\hat{G}_d$. Further denote the reduced dual of $G_d$ by $\hat{G}_d$, the set of $\pi \in \hat{G}_d$ which are weakly contained in the left regular representa-


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tion of $G_d$ on $l^2(G_d)$. Observe that $M_d(G)$ can be identified with $M(G_d)$, and $\mathfrak{M}_d(G) \cong \mathfrak{M}(G_d)$.

**Theorem 1.** There is a unique $C^*$-homomorphism of $\mathfrak{M}(\hat{G})$ onto $\mathfrak{M}(\hat{G}_d)$ such that for $\mu \in M(G)$, $P\mu$ is the discrete part of $\mu$, and kernel $P \supset \mathfrak{M}(\hat{G})$.

**Corollary 2.** For $\mu \in M_d(G)$, $\|\mu\|_d \leq \|\mu\|_r \leq \|\mu\|_\infty$; and thus $\hat{G}_d \supset \hat{G}_r$.

**Corollary 3.** If $G$ is nondiscrete and $\pi \in \hat{G}_d$, then $\pi \circ P$ is an irreducible representation of $\mathfrak{M}(\hat{G})$ and $\pi \circ P \in \kappa \hat{G} \setminus \hat{G}$. Further the map $\pi \to \pi \circ P$ is a homeomorphism of $\hat{G}_d$ into $\kappa \hat{G} \setminus \hat{G}$.

Let $G$ be nondiscrete and $S \subset \hat{G}$. Then define a seminorm on $\mathfrak{M}(\hat{G})$, called $S$-lim sup, to be the quotient norm of $\mathfrak{M}(S)/\mathfrak{M}_a(S)$. Recall $\mathfrak{M}(S) = \mathfrak{M}(\hat{G})/\mathfrak{M}(S)$ and $\mathfrak{M}_a(S) = \mathfrak{M}_a(\hat{G})/(\mathfrak{M}(S) \cap \mathfrak{M}_a(\hat{G}))$. If $G$ is compact or abelian then the $\hat{G}$-lim sup is identical to lim sup $\tau_{-\omega}|\tau(\phi)| = \inf_K \{\sup |\tau(\phi)|, \tau \in K\}$, $K$ a compact subset of $\hat{G}$, for $\phi \in \mathfrak{M}(\hat{G})$.

A locally compact group $G$ is said to be amenable if there exists a left invariant mean on the space of bounded continuous functions. Equivalent characterizations are that $\hat{G} = \hat{G}_r$, or that the representation $G \to \{1\}$ is in $\hat{G}_r$.

Under the assumption that $G_d$ is amenable, we can prove direct extensions of certain abelian-case theorems.

**Theorem 4.** If $G_d$ is amenable, $\phi \in \mathfrak{M}_d(\hat{G})$, then $\hat{G}$-lim sup $\phi$ = $\|\phi\|_\infty$. Further if $\mu \in M(G)$, then $\|\mu\|_\infty \geq \hat{G}$-lim sup $\mu \geq P\mu$.

**Theorem 5.** If $G$ is amenable, then $\mathfrak{M}(\hat{G}) = \mathfrak{M}_c(\hat{G}) \oplus \mathfrak{M}_d(\hat{G})$, where $\mathfrak{M}_c(\hat{G})$ is the closure in $\mathfrak{M}(\hat{G})$ of the set of continuous measures in $M(G)$.

**Corollary 6.** If $G$ is amenable and $\mu \in M(G)$ and $\mu \in \mathfrak{M}_d(\hat{G})$ then $\mu \in M_d(G)$.

If $G_d$ is amenable, then Corollary 2 reduces to: for $\mu \in M_d(G)$, $\|\mu\|_d = \|\mu\|_r = \|\mu\|_\infty$. This fact has been shown by Zeller-Meier in [2].

**Bibliography**


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