BOUNDARY BEHAVIOR OF HARMONIC FUNCTIONS ON HERMITIAN HYPERBOLIC SPACE

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Let \( D = \{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : h(z) = \text{Im} z_1 - \sum_2^n |z_k|^2 > 0 \} \), and \( B = \partial D = \{ z : h(z) = 0 \} \). Writing \( z_j = x_j + iy_j \) we let \( \beta \) be the measure on \( B \) given by \( d\beta = dx_1 dx_2 dy_2 \cdots dx_n dy_n \). \( D \) is a Siegel domain of Type II which is the image of the unit ball \( D = \{ z \in \mathbb{C}^n : \sum_1^n |z_k|^2 < 1 \} \) under the generalized Cayley transform:

\[
\begin{align*}
    z_1 &\mapsto i \frac{1 + z_1}{1 - z_1}, \\
    z_k &\mapsto i \frac{z_k}{1 - z_1}, \quad k = 2, \ldots, n.
\end{align*}
\]

Let \( N \) be the group of holomorphic automorphisms of \( D \) consisting of the elements \( (a, c) \in \mathbb{R} \times \mathbb{C}^{n-1} \) acting on \( D \) in the following way:

\[
(a, c) : z_k \mapsto z_k + a_1 \sum_{k=2}^n z_k \bar{c}_k + i \sum_{k=2}^n 2z_k \bar{c}_k,
\]

\[
(a, c) : z_k \mapsto z_k + c_k, \quad k \geq 2.
\]

\( N \) acts simply transitively on \( B \). We will consider real-valued functions on \( D \) which are harmonic with respect to the Laplace-Beltrami operator:

\[
L = h(z) \left\{ 4y_1 \frac{\partial^2}{\partial y_1 \partial \bar{y}_1} + \sum_2^n \frac{\partial^2}{\partial z_k \partial \bar{z}_k} + 2i \sum_2^n \frac{\partial^2}{\partial z_1 \partial \bar{z}_k} - 2i \sum_2^n \frac{\partial^2}{\partial \bar{z}_1 \partial \bar{z}_k} \right\}.
\]

In [2] Körányi defined the following notion of admissible convergence in \( D \): let us call

\[
\Gamma_a(u) = \left\{ z \in D : \text{Max} \left[ | \text{Re} z_1 - \text{Re} u_1 |, \sum_2^n | z_k - u_k |^2 \right] < a h(z), \quad h(z) < 1 \right\}
\]

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a truncated admissible domain of aperture $\alpha$ at $u \in B$. We say that $f$ on $D$ converges admissibly at $u$ to $l$ if $\lim_{z \to u; z \in \Gamma_{a}(u)} f(z) = l$, for some $\alpha > 0$.

The principal result of this note is the Theorem below, which is the analogue of results of Marcinkiewicz and Zygmund [3], Spencer [4], Calderón [1], and Stein [5]. (This is often referred to as the Area theorem for harmonic functions.) Let

$$\nabla f = \left(2h^{1/2} \frac{\partial f}{\partial z_1}, 2iz_2 \frac{\partial f}{\partial z_1}, \cdots, 2iz_n \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_n}\right)$$

and

$$|\nabla f|^2 = 4h \left| \frac{\partial f}{\partial z_1} \right|^2 + \sum_{k=2}^{n} \left| 2iz_k \frac{\partial f}{\partial z_1} + \frac{\partial f}{\partial z_k} \right|^2.$$ 

Let $E$ be a measurable set in $B$ and suppose that $f$ is a real-valued harmonic function in $D$.

**Theorem.** (a) If $f$ is admissibly bounded for each point of $E$ then

$$\int_{\Gamma_{a}(u)} h(z)^{-a} |\nabla f|^2 d\mu(z) < \infty$$

for almost every $u$ in $E$ and $\alpha > 0$, where $d\mu$ is Lebesgue measure.

(b) If, for each point $u$ of $E$, we can find an $\alpha$ such that the integral (1) is finite, then $f$ converges admissibly at almost every point of $E$.

The general outline of the proof follows Stein [5]. The differences arise from the fact that the Laplace-Beltrami operator is not uniformly elliptic. We first indicate how part (a) is proved. By a standard argument (see Calderón [1]) we may assume that $E$ is compact, and $f$ is uniformly bounded in $\Gamma_{a}(u)$, for $\alpha$ fixed, and all $u \in E$.

**Lemma 1.** If $f$ is bounded and harmonic in $\Gamma_{a}(0)$, then $h(z) |\partial f/\partial z_1|$ and $h(z)^{1/2} |\partial f/\partial z_k|$, $k \geq 2$, are bounded in $\Gamma_{a'}(0)$ for $\alpha' < \alpha$.

This result can be proved by using the Poisson integral representation for functions defined on images of spheres under the Cayley transform.

Let $\omega_{a}(E) = \bigcup_{\omega \in \mathcal{B}} \Gamma_{a}(u)$. We construct regions approximating $\omega_{a}(E)$. Write $z \in D$ as $z = [x, \bar{z}]$, where $x = x_1, \bar{z} = (z_2, \cdots, z_n)$, $t = h(z)$. Since $E$ is compact, $E_{t} = \{ [x, \bar{z}] : [x, \bar{z}]_{0} \in E \}$ is compact. For $0 < t < 1$ let $\Gamma_{a}(u)_{t} = \{ [x, \bar{z}]_{r+t^{2}} : [x, \bar{z}]_{0} \in \Gamma_{a}(u) \}$ and $r + t^{2} < 1$. Then $\{ \Gamma_{a}(u)_{t} \cap E_{t} \}_{u \in \mathcal{B}}$ forms an open cover of $E_{t}$. Choose a finite subcover for $t = t_{0} < 1$ and then for each $t < t_{0}$ choose one in the following manner: if $u_1, \cdots, u_{k(t)}$ are the base points chosen for the cover of
and if \( t' < t'' < t_0 \), then \( \{ u_1, \ldots, u_{k(t')} \} \supset \{ u_1, \ldots, u_{k(t''')} \} \). Let 
\[ \omega_i = \bigcup_{j=1}^{k(t)} \Gamma_\alpha(u_j). \]

**Lemma 2.** \( \int_{\omega_a(z)} |\nabla f| \, d\mu(z) < \infty. \)

We prove this by first applying Green's theorem to \( \omega_t \). Then, using the estimates of Lemma 1 translated by the group \( N \) and the uniform boundedness of \( f \), we obtain \( \int_{\omega_t} |\nabla f| \, d\mu(z) \leq k \int_{\omega_a} ds \) when \( k \) is independent of \( t \). Now we let \( t \) tend to 0, and observe that \( \int_{\omega_a} ds \leq M \) independently of \( t \). Part (a) then follows from:

**Lemma 3.** Suppose \( E \subset B \) is compact and \( f \) is locally bounded and positive in \( D \). If \( \int_{\omega_a} f \, d\mu < \infty \), then \( \int_{\Gamma_a(z)} h(z)^{-\alpha} f(z) \, d\mu(z) < \infty \) for all \( \beta > 0 \) and almost every \( u \in E \).

We now outline the proof of part (b).

**Lemma 4.** If \( \int_{\Gamma_a(0)} h(z)^{-\alpha} |\nabla f| \, d\mu(z) < \infty \), then \( h(z) |\partial f/\partial z_1| \) and \( h(z)^{1/2} |\partial f/\partial z_k| \), \( k \geq 2 \), are bounded in \( \Gamma_\alpha'(0) \) for \( \alpha' < \alpha \).

To prove this let
\[
D_1 = \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_1},
\]
\[
D_k = 2iz_k \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_k},
\]
\[
D_{k'} = -2iz_k \frac{\partial}{\partial z_1} + \frac{\partial}{\partial \bar{z}_k},
\]
\[
D_0 = z_1 \frac{\partial}{\partial z_1} + \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} + \frac{1}{2} \sum_{j=2}^{n} \left( z_k \frac{\partial}{\partial z_k} + \bar{z}_k \frac{\partial}{\partial \bar{z}_k} \right).
\]

We then observe that if \( f \) is harmonic then \( D_0f, D_1f, D_2f, D_kf \) are harmonic, and thus can be represented as Poisson integrals. Now \( |\nabla f|^2 \) dominates \( h |D_1f|^2, |D_2f|^2, |D_kf|^2 \) and \( h^{-1} |D_0|^2 \) in \( \Gamma_\alpha(0) \); and the latter dominate \( h |\partial f/\partial z_1|^2 \) and \( |\partial f/\partial z_k|^2 \) for \( k \geq 2 \), in \( \Gamma_\alpha(0) \). Now, using Green's theorem and Lemma 4, we have
\[
\int_{\partial \omega_1} f^2 \, ds \leq k \int_{\partial \omega_1} |f| \, ds + k' \int_{\partial \omega_1} |\nabla f|^2 \, d\mu.
\]

**Lemma 5.** Suppose \( E \subset B \) is compact, \( f \) is nonnegative and locally bounded in \( D \), and for each \( u \in E \), there exists an \( \alpha > 0 \) such that \( \int_{\Gamma_a(u)} f \, d\mu < \infty \). Then for every \( \epsilon > 0 \) and \( \beta > 0 \) there exists a compact set \( F \subset E \) such that \( \operatorname{meas}(E \setminus F) < \epsilon \), and \( \int_{\omega_{a}(0)} h(z)^{-\alpha} f(z) \, d\mu(z) < \infty \).
Applying this to the inequality above we have \( \int_{\partial \Omega} |f|^2 \, ds \leq M \) independently of \( t \). Now a standard argument (see Stein [5]) shows that 
\[ |f(z)| \leq cg(z) + c' \]
in \( \omega_\alpha(E) \) where \( g \) is the Poisson integral of some function in \( L^2(B) \). The result now follows from Korányi [2].

References