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this issue.

A REMARK ON CLASSIFICATION OF RIEMANN
SURFACES WITH RESPECT TO Δu = Pu

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1. Consider a $C^1$ differential $P(z)\, dx\, dy$ ($z=x+iy; \, P \geq 0, \, P \neq 0$) on an open Riemann surface $R$. We denote by $\mathfrak{O}_{PX}$ the set of pairs $(R, P)$ such that the subspace $PX(R)$ of the space $P(R)$ of $C^2$ solu­
tions $u$ of $\Delta u = Pu$ on $R$ determined by a property $X$ reduces to $\{0\}$. Here the possibilities for $X$ that we consider are B (boundedness), D (Dirichlet-finite: $D_R(u) = \int_{\mathcal{R}} |\nabla u(z)|^2 \, dx\, dy$), E (energy-finite: $E_R(u) = D_R(u) + \int_{\mathcal{R}} P(z)\,(u(z))^2 \, dx\, dy$), and their combinations BD, BE. The purpose of this note is to announce that the following very
simple pair $(U, Q)$ given by

(1) $U = \{z; \, |z| < 1\}$, \quad $Q(z) = (1 - |z|)^{-1}$

is an example of the strict inclusion relation

(2) $\mathfrak{O}_{PD} < \mathfrak{O}_{PE}$

Here and hereafter $\mathfrak{A} < \mathfrak{B}$ means that $\mathfrak{A}$ is a proper subset of $\mathfrak{B}$. This
type of classification problem for Riemann surfaces proposed by

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Dirichlet integral, energy integral, classification theory of Riemann surfaces, Little­wood theorem.

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Ozawa [5] and Royden [6] thus comes to the following complete conclusion:

\[(3) \quad \Theta_G < \Theta_{PB} < \Theta_{PD} = \Theta_{PB}D < \Theta_ME = \Theta_{PB}E\]

for pairs \((R, P)\) of Riemann surfaces \(R\) and \(C^1\) differentials \(P\) on \(R\) \((P \geq 0, P \neq 0)\). Here \(\Theta_G\) is the set of pairs \((R, P)\) such that \(R\) does not carry a harmonic Green's function. For the latest information on this subject, refer to [1].

2. Since the harmonic Green's function \(G(z, \xi)\) on \(U\) is \(\log(1 - |z|/|\xi|)\), we have

\[\int_0^{2\pi} G(re^{i\theta}, \xi) \, d\theta = -2\pi \max(\log r, \log |\xi|).\]

By virtue of this relation it is easy to evaluate

\[(D) \quad \int_{U \times U} G(z, \xi)Q(z)Q(\xi) \, dx \, dy \, d\xi \, d\eta < \infty \quad (\xi = \xi + i\eta).\]

By the integral comparison theorem ([3], [4], [2]), (D) implies the existence of an order-preserving isometric vector space isomorphism \(u \mapsto \tau u\) of \(QBD(U)\) onto \(HBD(U)\) (H stands for harmonic) determined by

\[(4) \quad u = \tau u - \frac{1}{2\pi} \int_U G(\cdot, \xi)Q(\xi)u(\xi) \, d\xi \, d\eta.\]

In particular we obtain \((U, Q) \in \Theta_{PBD}.\) Since \(\Theta_{PB}D = \Theta_{PD}\) ([2], [3]), we conclude that

\[(5) \quad (U, Q) \in \Theta_{PD}.\]

Observe that every \(u \in QBE(U)\) \((\subset QBD(U))\) is a difference of two nonnegative \(u_1\) \(\in\) \(QBE(U)\), i.e. \(u = u_1 - u_2\) (Royden [6]). Let \(u \in QBE(U)\) and \(u \geq 0.\) Since

\[\int_{|z|<1} (1 - |z|) \, d\mu(z) < \infty \quad (d\mu(z) = Q(z)u(z) \, dx \, dy \geq 0),\]

by Littlewood's theorem we have

\[\lim_{r \to 1} \int_U G(re^{i\theta}, \xi) \, d\mu(\xi) = 0\]

for almost every \(\theta \in [0, 2\pi]\) (cf. e.g. Tsuji [7, p. 170]). As the bounded harmonic function \(\tau u\) has the radial limit \(\lim_{r \to 1} \tau u(re^{i\theta})\) for almost
every $\theta \in [0, 2\pi]$, we see by (4) that the same is true for $u$ and a fortiori

$$\lim_{r \to 1} u(re^{i\theta}) = \lim_{r \to 1} \tau u(re^{i\theta}) = u^*(\theta) \geq 0$$

almost everywhere on $[0, 2\pi]$. If $u^*(\theta) > 0$, then $u(re^{i\theta}) > u^*(\theta)/2$ for $0 < \epsilon < r < 1$ and therefore

$$l(\theta) = \int_0^1 Q(re^{i\theta})(u(re^{i\theta}))^2 r \, dr$$

$$\geq 4^{-1}(u^*(\theta))^2 \int_{\epsilon}^1 \frac{1}{1-r} \, dr = \infty.$$ 

By Fubini's theorem,

$$\int_U Q(z)(u(z))^2 \, dx \, dy = \int_0^{2\pi} l(\theta) \, d\theta.$$ 

In view of (7), the quantity (8) is finite only if $u^*(\theta) = 0$ almost everywhere on $[0, 2\pi]$. Then by Poisson's representation, we deduce $\tau u \equiv 0$ and consequently by $0 \leq u \leq \tau u$ we conclude that $u \equiv 0$. Therefore $u \in QBE(U)$ ($u \geq 0$) implies that $u \equiv 0$, i.e. $(U, Q) \in \Theta_{PBE}$. Since $\Theta_{PBE} = \Theta_{PE}$ (Royden [6]), we obtain

$$\tag{9} (U, Q) \in \Theta_{PE}.$$ 

The relations (5) and (9) imply (2).

3. In our recent paper [4] (see also [1]) we determined the degeneracy character of $(E_m, P_a)$ ($E_m$: $m$-dimensional Euclidean space ($m \geq 3$); $P_a(x) \sim |x|^{-\alpha}$ ($|x| \to \infty$)) as follows:

$$\tag{10} \begin{align*}
(E_m, P_a) &\in \Theta_{PB} - \Theta_G \quad (\alpha \leq 2); \\
(E_m, P_a) &\in \Theta_{PD} - \Theta_{PB} \quad (2 < \alpha \leq (m + 2)/2); \\
(E_m, P_a) &\in \Theta_{PE} - \Theta_{PD} \quad ((m + 2)/2 < \alpha \leq m); \\
(E_m, P_a) &\in \Theta_{PE} \quad (m < \alpha).
\end{align*}$$

The 2-dimensional analogue is $(U, P_a)$ ($P_a(z) \sim (1 - |z|)^{-\alpha}$ ($|z| \to 1$)): The pair $(U, P_a)$ will be an example for each strict inclusion in (3) if $\alpha$ is properly chosen, which will be discussed in detail elsewhere.

ADDED IN PROOF. The 2-dimensional analogue of (10) for $(U, P_a)$ is:

$$2 \leq \alpha; \ 3/2 \leq \alpha < 2; \ 1 \leq \alpha < 3/2; \ \alpha < 1$$

(M. Nakai, The equation $\Delta u = Pu$ on the unit disk with almost rotation free $P \geq 0$ (to appear)).
REFERENCES


4. ———, The equation $\Delta u = Pu$ on $E^n$ with almost rotation free $P \geq 0$, Tôhoku Math. J. (to appear).


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