1. Introduction. In [1], S. P. Wang uses the techniques of [2] to prove a converse to a Selberg lemma for solvable groups. In [3], the author gave an elementary proof of the main result of [2] using the semisimple splitting. It is quite natural to expect that the results of [1] should also have an elementary proof in terms of the semisimple splitting. We do so in this paper.

2. Preliminaries. Let \( S \) be a simply connected solvable analytic group with nil-radical \( H \). Suppose we imbed \( S \) as a subgroup of \( \text{GL}(n, \mathbb{R}) \), the group of all \( n \times n \) nonsingular real matrices. Let \( \overline{\alpha}(S) \) denote the algebraic hull of \( S \). Then we can write \( \overline{\alpha}(S) = N \cdot T \), where \( N \) is the group of all unipotent matrices in \( \overline{\alpha}(S) \) and \( T \) is a maximal abelian subgroup of semisimple matrices in \( \overline{\alpha}(S) \). The relevant facts about algebraic groups can be found in [6]. In [3] we stated the following result primarily due to L. Auslander [4].

There exists an imbedding of \( S \) as a subgroup of \( \text{GL}(n, \mathbb{R}) \) satisfying the following properties.

1. \( H \subset N \).
2. The projection mapping \( P: \overline{\alpha}(S) \to N \) restricted to \( S \) defines a diffeomorphism of \( S \) onto \( N \). We denoted the restriction of \( P \) to \( S \) by \( n:S \to N \).

Denote the projection mapping of \( \overline{\alpha}(S) \) into \( T \) by \( t \). Let \( C \) be a closed subgroup of \( S \). As we have seen in [3] we can choose \( T \) so that \( t(C) \subset \overline{\alpha}(C) \). Let \( C' = \overline{\alpha}(C \cap H)C \). Then \( C'/C \) is compact and \( C' \) is closed in \( S \). From our choice of \( T \) and since \( [\overline{\alpha}(C), \overline{\alpha}(C)] \subset \overline{\alpha}(C \cap H) \) it follows that \( n(C') \) is a closed subgroup of \( N \). It is easy to see that the following statements are equivalent.

1. \( S/C \) is compact.
2. \( S/C' \) is compact.
3. \( N/n(C') \) is compact.
4. \( N = \overline{\alpha}(n(C')) \).


Key words and phrases. Solvable, nilpotent, Lie group, topological group, algebraic hull, matrix groups, unipotent, semisimple splitting.

Copyright © 1971, American Mathematical Society
3. **Wang's Theorem A.** We shall assume the following two simple lemmas from [1].

**Lemma 3.1.** Let $V$ be a finite-dimensional vector space over the reals, $W$ a proper subspace of $V$ and $G$ a connected solvable subgroup of $GL(V)$. Then there is a neighborhood $\Omega$ of the identity in $G$ such that $\bigcup_{g \in \Omega} g(W) \neq V$.

**Lemma 3.2.** Let $X$ be a nonempty conic open subset of $V$ and $\Omega$ a compact neighborhood of 0 in $V$. Then for every $x$ in $X$, there is a positive number $r$ such that $sx + \Omega \subset X$ for all $s \geq r$.

Let $C$ be a closed subgroup of the simply connected solvable analytic group $S$. We shall assume the notation and conventions of §2.

We say that $C$ has the Selberg property in $S$ if and only if for any $s$ in $S$ and for any neighborhood $\Omega$ of 1 in $S$ there exists $u$ and $v$ in $\Omega$ and an integer $l > 0$ such that $us^l v$ is in $C$.

**Theorem A.** Suppose that $C$ has the Selberg property in $S$. Then $S/C$ is compact.

**Proof.** By 1.2 of [5], $C'$ has the Selberg property. Since $S/C$ is compact if and only if $S/C'$ is compact we can replace $C$ by $C'$ in our discussion. Thus without loss of generality we will assume throughout that the subgroup $C$ under consideration has the addition property that $\alpha(C \cap H) \subset C$. It follows that $n(C)$ is a subgroup of $N$. We must prove that $N = \alpha(n(C))$.

Assume that $N$ is abelian. Suppose that $N \neq \alpha(n(C))$. Let $\Omega_1$ be the set of all $x$ in $N$ such the euclidean absolute value of $x$ is less than one. Let $\Omega_2$ be the set of all $x$ in $N$ whose euclidean absolute value is less than one half. Since $S$ normalizes $N$, by Lemma 1 it is easy to see that there is a compact symmetric neighborhood $\Omega$ of 1 in $S$ such that

(a) $\bigcup_{u \in \Omega} u \alpha(n(C)) u^{-1} \neq N$.

It follows from elementary topological group theory techniques (see p. 95 of [7]) that $\Omega$ can be chosen with the addition properties that

(b) For all $u$ in $\Omega$, $u \Omega_2 u^{-1} \subset \Omega_1$.

(c) $\Omega^2 \subset n^{-1}(\Omega_2)$ where $n$ is the homeomorphism of $S$ onto $N$ introduced in §2.

By [5], $HC/H$ has the Selberg property in the vector group $S/H$. Thus by [5] again we have that $S/HC$ is compact. It follows from our previous discussion that $N = H + \alpha(n(C))$. 


Let $X = N - \bigcup_{u \in \Omega} u \alpha(n(C))u^{-1}$. Then $X$ is a nonempty open conic. It follows that there is an $h$ in $H$ such that $h^l + \Omega \subset X$, for all positive integers $l$. Since $C$ has the Selberg property in $S$ there are elements $u$ and $v$ in $\Omega$ and a positive integer $l$ such that $uh^lv$ is in $C$. Thus $uh^l n(uv)$ is in $n(C)$. Note that by $n(uv)$ we mean the application of the projection map $n$ to the product $uv$. From this equation we get that $h^l n(uv)u$ is in $u^{-1}n(C)u \subset \Omega$. Thus $h^l + \Omega \subset X$, a contradiction.

Suppose $N$ is not abelian. Let $[x, y]$ denote the commutator $xyx^{-1}y^{-1}$ of $x$ and $y$. Denote the last nontrivial term in the lower central series of $N$ by $M$. Using induction on the number of terms of the lower central series of $N$ we can assume that the theorem holds in the group $S/M$. By [5], $MC/M$ has the Selberg property in $S/M$. Thus $N = M + \alpha(n(C))$.

Let $z$ and $z'$ be in $N$. We can write $z = xy$ and $z' = x'y'$ where $x$ and $x'$ are in $M$ and $y$ and $y'$ are in $\alpha(n(C))$. Since $M$ is central in $N$, $[z, z'] = [y, y']$ is in $\alpha(n(C))$. Thus $MC[N, N] \subset \alpha(n(C))$.

4. **Wang's Theorem B.** We shall be satisfied with proving Wang's Theorem B for the special case of $S$ simply connected.

**Theorem B.** Let $S$ be a simply connected solvable analytic group, $C$ a discrete subgroup of $S$ such that $S/C$ is compact, and $Z$ the centralizer of $C$ in $S$. Then $Z$ is abelian.

**Proof.** Since $Z$ commutes with $C$ it commutes with $\alpha(C)$. Thus $Z$ acts by the identity map on $N$. This easily implies that $Z$ must be contained in $N$. Thus $Z$ is abelian.

**References**


Yale University, New Haven, Connecticut 06520