cussion of the French school of permutation analysis, such as the Cartier-Foata theory of permutations of a multiset, the work of A. Jacques on planar graphs and symmetric groups, and the Foata-Schützenberger theory of Eulerian numbers.

Chapter 5 is devoted to the famous Pólya theorem on enumeration under group action, including the generalization due to deBruijn. Besides the usual applications to counting graphs, coloring cubes, etc., an unusual application is given to the enumeration of knots. There is a minor error on p. 170—$S_n \otimes S_n$ is not the group connected with directed graphs.

The above survey of topics points out the magnificent job Berge has done in sifting out from the vast literature of combinatorics the most interesting, elegant and important results connected with enumeration. Anyone who reads this books will not only derive many hours of fascination and enjoyment, but will also have a much better grasp of the meaning of the current combinatorial revolution. To paraphrase from the Foreword, Berge's book will go a long way toward unknotting the reader from the tentacles of the Continuum and inducing him to join the Rebel Army of the Discrete.

RICHARD STANLEY

*Modular Lie algebras* by G. B. Seligman. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 40. Springer-Verlag Inc., New York, 1967. ix+165 pp. $9.75.

This book is the first to be devoted to Lie algebras over fields of characteristic $p>0$, the so-called modular Lie algebras of the title. Other recent books, such as Jacobson's *Lie Algebras*, are concerned with Lie algebras over an arbitrary field to the extent to which the theory for characteristic 0 may be generalized to arbitrary fields. However, there are significant differences between Lie algebras of characteristic 0 and those of characteristic $p>0$. This is the first book in which the latter are studied in a systematic way.

Complex and real Lie algebras, because of their use in the study of Lie groups, comprise a classical subject with which many mathematicians are acquainted. The extension of classical results to Lie algebras over an arbitrary field began in the 1930's and contributed to the further development of the theory. Crucial differences between Lie algebras of characteristic 0 and of characteristic $p>0$ were recognized early. In the late thirties and early forties a number of significant papers by Jacobson, Zassenhaus and others on Lie algebras of prime characteristic were published. But the difficulties encountered in the study of these algebras appeared intractable, and there was only sporadic progress until in the mid-fifties there was an explosion of exciting results (sparked perhaps by the Society's first Summer Mathematical Institute in 1953). A high level of interest and accomplishment has been maintained since that time. About half of this book is devoted, with fairly complete proofs, to the theory of the so-called "classical Lie algebras" (invented by the author and W. H. Mills). The other half consists of an exhaustive survey of modular Lie algebras, essentially without proofs but with numerous references to the work of many mathematicians (of whom it would be invidious to list less than a dozen).

In the first chapter the author summarizes efficiently such fundamentals for the study of Lie algebras over an arbitrary field as representations, universal associative algebras, the Poincaré-Birkhoff-Witt theorem, free Lie algebras, nilpotent Lie algebras, Engel's theorem, Cartan subalgebras, the Killing form, trace forms, derivations, and extension of the base ring. Characteristically, the author states Ado's theorem with only a reference to the proof, but actually proves Iwasawa's analogue for the modular case: any finite-dimensional Lie algebra of characteristic $p>0$ has a faithful finite-dimensional representation. As is well known, the following three propositions are equivalent for finite-dimensional Lie algebras $\mathfrak{L}$ of characteristic 0, and characterize semisimple Lie algebras of characteristic 0:

(A) $\mathfrak{L}$ has nondegenerate Killing form.

(B) $\mathfrak{L}$ is a direct sum of ideals which are simple Lie algebras.

(C) The only abelian ideal of $\mathfrak{L}$ is 0.

The author proves that, over an arbitrary field, each of the above propositions implies the next. That each of the reverse implications fails for modular Lie algebras is the source of much of the great difficulty, and also much of the interest, in the study of these algebras. The author takes (C) as his definition of semisimplicity. Classification of the algebras in (B) is an open problem, even for algebraically closed fields. Modification of (A) gives the "classical Lie algebras" of the next three chapters.

In the first chapter the author also introduces a notion which can be defined only for characteristic $p>0$; namely, the important concept of a restricted Lie algebra (=Lie $p$-algebra). This is a Lie algebra $\mathfrak{L}$ over a field $\mathfrak{F}$ of characteristic $p>0$, together with a formal $p$th power mapping $z\to z^{[p]}$ in $\mathfrak{L}$ which satisfies formal properties inherited from the $p$th power mapping $x\to x^p$ in a universal associative algebra. The assumption that one is dealing with a restricted Lie algebra, together with use of the associated concepts of restricted representation and restricted universal associative algebra, is sometimes enough to introduce order into an otherwise chaotic situation.

The second chapter is devoted to the determination of the classical semisimple Lie algebras. Let $\mathfrak{F}$ be a field of characteristic $\neq 2, 3$. The author calls a finite-dimensional Lie algebra $\mathfrak{L}$ over $\mathfrak{F}$ classical if:

(i) the center of $\mathfrak{L}$ is 0;

(ii) $[\mathfrak{L}\ \mathfrak{L}]=\mathfrak{L}$;

(iii) $\mathfrak{L}$ has an abelian Cartan subalgebra $\mathfrak{H}$, called a classical Cartan subalgebra, relative to which:

(a) $\mathfrak{L}=\sum \mathfrak{L}_\alpha$, where $[x\ h]=\alpha(h)x$ for all $x\in L_\alpha$, $h\in\mathfrak{H}$;

(b) if $\alpha\neq 0$ is a root, $[\mathfrak{L}_\alpha\ \mathfrak{L}_{-\alpha}]$ is one-dimensional;

(c) if $\alpha$ and $\beta$ are roots, and if $\beta\neq 0$, then not all $\alpha+k\beta$ are roots.

One recognizes that every split semisimple Lie algebra of characteristic 0 is classical. The axioms (i)–(iii) have been chosen as being precisely sufficient to give the structure which is familiar in the characteristic 0 case: a classical Lie algebra is a direct sum of simple ideals which may be classified into four infinite classes A–D and five exceptional algebras $G_2$, $F_4$, $E_6$, $E_7$, $E_8$.

The tools employed to achieve this classification are the Cartan decomposition, split 3-dimensional algebras, strings of roots, Cartan integers, fundamental root systems, components in a fundamental root system, existence of isomorphisms, the Weyl group, and the Chevalley basis. No proof is given for the latter (which is used to prove the existence of the classical algebras) but the other concepts are developed in great detail. At the end of this chapter the advantage of the axioms (i)–(iii) over the assumption of a nondegenerate Killing form is explained, and several generalizations of the theory are summarized.

Automorphisms of classical Lie algebras are studied in the third chapter. It would be futile to attempt to improve on the historical survey and the summary of this chapter which the author gives as an introduction to it, so we quote from the book (except for deleting references) as follows: "The automorphism groups of classical Lie algebras, in the sense of the previous chapter, have been studied for the four 'great classes' A–D by Jacobson, considering the most natural realizations of these algebras. A unified approach has been made by the author, substituting certain combinations of algebraic operations for the exponential functions used in the fundamental work of Gantmacher in the complex case. Where the author's results are incomplete (in case the ground field is not algebraically closed), they have been completed by Steinberg. Indeed, Steinberg is able to deal with characteristics 2 and 3 as well, since he obtains his Lie algebras by Chevalley's process from a complex semisimple Lie algebra. We reproduce here the results of Steinberg, restricted to the case of classical algebras in our sense, as well as giving essentially

Chevalley's results on the general structure of the groups of Chevalley, when regarded as subgroups of the automorphism groups of classical algebras. Finally, we give interpretations for these results in terms of the natural realizations for types A–D, as well as for the exceptional algebras."

The author gives a masterly exposition (for characteristic $\neq 2, 3$) of this material which is of fundamental importance for anyone interested in the theory of simple groups. A detailed description is given of Chevalley groups and the Bruhat-Chevalley decomposition, and conjugacy of Cartan subalgebras is proved. The chapter concludes with a section on realizations of the classical simple Lie algebras. The realizations for the infinite classes A–D are the familiar special linear, orthogonal and symplectic Lie algebras (except that for $A_n$, if the characteristic $p$ divides $n+1$, one must factor out a 1-dimensional center). The exceptional algebra $G_2$ is realized as the derivation algebra of the split Cayley algebra. The exceptional algebras $F_4$, $E_6$, $E_7$ have realizations in terms of the split exceptional central simple Jordan algebra, while the treatment of the exceptional algebra $E_8$ is unavoidably sketchy. The automorphism groups of the classical simple Lie algebras are given in terms of these realizations.

Let $\mathfrak{K}$ be a field, and $\mathfrak{F}$ a subfield of $\mathfrak{K}$. If $\mathfrak{A}$ is a linear algebra over $\mathfrak{K}$, an $\mathfrak{F}$-form of $\mathfrak{A}$ is a linear algebra $\mathfrak{B}$ over $\mathfrak{F}$ such that $\mathfrak{B}_{\mathfrak{K}}$ is $\mathfrak{K}$-isomorphic to $\mathfrak{A}$. The fourth chapter of this book is devoted to a study of forms of the classical Lie algebras. All Lie algebras $\mathfrak{L}$ with nondegenerate Killing form over an arbitrary field $\mathfrak{F}$ of characteristic $\neq 2, 3$ may be obtained in terms of forms of classical simple Lie algebras. For the reader who is acquainted with the determination of central simple real Lie algebras from simple complex Lie algebras, the point is clear, although the complexity of the situation for an arbitrary field $\mathfrak{F}$ may not initially be apparent. For the determination of forms of a classical simple Lie algebra $\mathfrak{L}$ one uses the automorphisms of $\mathfrak{L}$ which were studied in the previous chapter.

Following some preliminaries concerning splitting fields, Galois semiautomorphisms and 1-cohomology, forms of algebras in the classes A–D (except for $D_4$, which is treated as an exceptional algebra) are determined in terms of simple involutorial associative algebras. Forms of $G_2$ (resp. $F_4$) are the derivation algebras of Cayley algebras (resp. exceptional central simple Jordan algebras). For $D_4$, $E_6$, $E_7$, $E_8$ the situation is more complicated, and there are still a number of open problems.

Clearly the $\mathfrak{F}$-forms of a classical simple Lie algebra $\mathfrak{L}$ depend upon the field $\mathfrak{F}$, and some of the complexities of the theory may disappear for algebras over special types of fields. The author describes the case of finite fields, where the forms of $\mathfrak{L}$ are known completely and are of particular interest in the study of finite groups. This chapter ends with a discussion of the automorphism groups of forms of classical simple Lie algebras.

The reader who is well acquainted with Lie algebras of characteristic 0 will find much that is familiar in the three chapters on classical Lie algebras. Perhaps some of the detailed information on realizations, automorphisms and forms, particularly for the exceptional algebras, will be new to him. However, the fifth chapter, a comparison of the modular and nonmodular cases, will be a complete revelation to most readers. Here the astonishing variety of things which can happen at characteristic $p>0$ is exposed. In a very well-organized way the author reveals the wildly disorderly character of modular Lie algebras as compared with those of characteristic 0.

Lie's theorem fails. The derived algebra of a solvable Lie algebra need not be nilpotent. The weights of a nilpotent Lie algebra of linear transformations need not be linear. Every modular Lie algebra has a faithful completely reducible representation and a faithful representation which fails to be completely reducible. (A restricted Lie algebra $\mathfrak{L}$ has all its restricted representations completely reducible if and only if $\mathfrak{L}$ is abelian and $x \rightarrow x^{[p]}$ is (1-1). Also there is some order in the theory of irreducible restricted representations of classical modular Lie algebras.) The tensor product of two completely reducible representations need not be completely reducible. If $\mathfrak{L}$ is a modular Lie algebra such that the first cohomology group $H^1(\mathfrak{L}, \mathfrak{M})=0$ for all $\mathfrak{L}$-modules $\mathfrak{M}$, then $\mathfrak{L}=0$. The second cohomology group $H^2(\mathfrak{L}, \mathfrak{M})$ need not be 0 even when $\mathfrak{L}$ is classical and $\mathfrak{M}$ is an irreducible restricted $\mathfrak{L}$-module.

By classifying classical simple Lie algebras and their forms, one knows all (finite-dimensional) simple Lie algebras of characteristic 0. At the present time one does not know all simple modular Lie algebras, not even all restricted ones. In the 1930's, in addition to the classical simple Lie algebras and some of their forms, only the Jacobson-Witt algebras were known. Let $\mathfrak{F}$ be a field of characteristic $p>0$, and $\mathfrak{A}_n$ be the group algebra over $\mathfrak{F}$ of the direct product of $n$ copies of the cyclic group of order $p$. The derivation algebra $\mathfrak{W}_n$ of $\mathfrak{A}_n$ is called the split Jacobson-Witt algebra of order $n$ over $F$; $\mathfrak{W}_n$ is a simple restricted Lie algebra unless $p=2$ and $n=1$. Every finite-dimensional restricted Lie algebra can be embedded in a split Jacobson-Witt algebra. The author lists all known classes of simple modular Lie algebras. Most of these were discovered in the 1950's and sixties. Some have realizations as certain distinguished subalgebras of $\mathfrak{W}_n$; others are defined by finite groups of functions. They are a disparate lot, and one does not have the feeling that they even begin to exhaust the possibilities.

This chapter, comparing the modular and nonmodular cases, continues with a detailed summary of results on derivations. Following a discussion of extension of the base field, the author then proves a number of theorems about Cartan subalgebras. For example, every solvable Lie algebra has a Cartan subalgebra, and every restricted Lie algebra has a Cartan subalgebra. The chapter concludes with proofs of a number of theorems concerning nilpotent elements and special subalgebras.

The sixth and final chapter is concerned with some ways in which modular Lie algebras arise in other parts of mathematics. There is a detailed discussion of nilpotent groups and Lie algebras, with application to the restricted Burnside problem. The author gives a succinct account of linear algebraic groups and their relationship to Lie algebras. Although the group-Lie algebra correspondence is not as good at characteristic $p>0$ as one might desire, a number of very satisfactory results have been obtained. Next the author summarizes the theory of formal Lie groups and hyperalgebras. After that he describes the relationships between purely inseparable extension fields of exponent one and derivations and, more generally, between purely inseparable extensions of arbitrary exponent and higher derivations. The chapter ends with a section on infinite-dimensional analogues of the classical Lie algebras.

This book is very carefully written in a lucid style. There are numerous references to an exhaustive bibliography. This reviewer noted only two trivial misprints, together with the omission from the bibliography of the papers of W. Landherr (although one of these papers is cited in the foreword for its historical importance).

Although this is the first book on modular Lie algebras, it promises to be the standard text until much more research is done. It serves both as an excellent reference book and as a thoughtfully organized introduction to a fascinating subject.

R. D. SCHAFER