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FROBENIUS RECIPROCITY IN ERGODIC THEORY

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1. Introduction. The analogy between group actions and group representations, which has guided algebraists' intuition since the turn of the century, has been slow in influencing analysis. To be sure, the analogy between "strict sense" and "wide sense" notions, first noticed in probability, can be traced to be a natural analog of the preceding analogy, and in fact has led to the discovery of a host of new results (for example by Nelson\(^1\) and Rota where the strict sense analogs of well-known Hilbert-space theorems of Nagy and Naïmark are worked out). More recently, a conjecture of Rota (1962) regarding a strict-sense analog of Schreiber's Theorem has been settled in the affirmative by McCabe and Shields, using Ornstein's deep results relating entropy to conjugacy. Despite this and much other work, however, a systematic "spectral theory in the strict sense" for ergodic trans-

\(^{1}\) Authors' names refer to the bibliography at the end. The results of the present announcement were first presented by Rota at the Symposium on Functional Analysis held at the U. S. Naval Postgraduate School in Monterey, California, in October 1969. We are grateful to Professor Carroll Wilde for permission to reproduce parts of Professor Rota's lecture outside the Proceedings of the Symposium. Professor Rota's work was carried out under O.N.R. Contract N00014-67-A-0204-0016 and Professor Ramsay's under NSF Grant GP 11622.


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formations of measure spaces remains to be developed. The far-reaching ideas recently introduced by Mackey, and arrived at by analogies between transitive actions of finite groups and ergodic actions of locally compact groups, offer renewed promise for this program. The notion of virtual group, introduced by Mackey with the purpose of developing an "action" analog of the notion of induced representation, has already found a multitude of applications (Lange and Ramsay) and, we surmise, will substantially influence and deflect the development of ergodic theory.

Our present purpose is to strengthen and develop one of Mackey's constructions by placing it in the context of categorical algebra. It has been known for some time (see e.g. Lang) that the induced representation functor is the left adjoint of the restriction (to a subgroup) functor. Guided by this analogy, we are led to surmise that the construction of an induced action should be, in a suitable context, an adjoint functor. This turns out to be the case (Theorem 2). To establish it, a detailed detour into the treacherous waters of pointwise measure theory seems necessary. Some of the basic spadework has been done by Mackey, but a few notions need to be retouched. For this and other reasons, we have found it prudent to give all definitions, some of which—in particular that of virtual group—slightly differ from Mackey's.

The present work points to several further developments, notably: a "strict sense" analog of the imprimitivity theorem; a corresponding "right adjoint" to the functor $R$; and a development of the functor $R$ and its left adjoint $M$, the "Mackey functor", along the lines of the theory of triples. We hope to consider them in forthcoming publications.

2. Definitions. Recall that a group is a small category with one object, where every morphism has an inverse. A groupoid is a small category where every morphism has an inverse.

We refer to Moore for the definitions of Borel space, Borel map, countably generated Borel space, analytic Borel space, equivalent $\sigma$-finite measures, measure class (denoted by $\mathcal{C}$). Let $T$ be a Borel map; if $T^{-1}(A)$ is a null set whenever $A$ is a null set then the measure class is invariant under $T$. An analytic Borel groupoid is a groupoid, together with an analytic Borel space structure on the set of morphisms, and a measure class $\mathcal{C}$ (this induces a similar structure on the set $U$ of objects of $V$ by identifying an object with the corresponding identity morphism), such that:

(a) The domain $D$ of composition of two morphisms is a Borel subset of $V \times V$, under the product Borel structure.
(b) Composition \((f, h) \rightarrow fh\) and inversion \(f \rightarrow f^{-1}\) are Borel maps of \(D(V)\) to \(V\), and inversion leaves invariant the measure class \(C\).

Let \(d\) and \(r\) be the (Borel) maps of a morphism to its "domain" and "range"; then \(d\) induces a measure class \(C\) on \(U\), setting \(C(\mu) = \mu(d^{-1}(A))\) for \(\mu\) in \(C\). Any \(\mu\) in \(C\) satisfies \(\mu = \int \mu_s \circ ds\), where \(\mu_s\) is a measure on \(V\) living on \(d^{-1}(s)\), and, for fixed Borel set \(A\) in \(V\), the function \(s \rightarrow \mu_s(A)\) is Borel on \(U\). The fibering is unique: if \(\mu = \int \nu_s \circ ds\), then \(\nu_s = \mu_s\) for \(C\)--almost all \(s\); furthermore, changing \(\mu\) to an equivalent measure does not change the measure class of \(\mu_s\) for \(C\)--almost all \(s\); this leads to measure classes \(C_s\) on almost all fibers \(d^{-1}(s)\).

(c) For \(s \in U\) and \(r(f) = s\), the map \(h \rightarrow hf\) carries \(d^{-1}(s)\) biuniquely to \(d^{-1}(d(f))\) and \(C_{r(f)}\) to \(C_{d(f)}\).

We require that \(C_{r(f)} = C_{d(f)}\) for all \(f\) with \(r(f)\) and \(d(f)\) in some co-null (complement of a null) Borel set \(U_0\) of \(U\). Say that \((V, C)\) is \(\text{ergodic}\) whenever every real Borel function \(\phi\) on \(U\) such that \(\phi(d(f)) = \phi(r(f))\) for almost all \(f\) in \(V\) is \(C\) a.e. constant. A \(\text{virtual group}\) is an \(\text{ergodic analytic Borel groupoid}\). In a virtual group \((V, C)\), let \(U_0\) be a co-null Borel set of \(U\). Taking all \(f \in V\) s.t. both \(d(f)\) and \(r(f)\) are in \(U_0\), we obtain another virtual group, the \(\text{inessential contraction} V|U_0\). Again, say that two objects \(u\) and \(v\) are \(\text{equivalent}\) when \(d(f) = u\) and \(r(f) = v\) for some morphism \(f\); if \(A \subseteq U\), write \([A]\) for the \(\text{satisfaction}\) of \(A\) under this equivalence relation; note that \([A]\) = \(r(d^{-1}(A))\) and that \([A]\) is naturally an analytic Borel space if \(A\) is a Borel set.

3. \textbf{Categories and functors.} A \(\text{strict homomorphism} \psi\) between virtual groups \((V_1, C_1)\) and \((V_2, C_2)\) is a functor from \(V_1\) to \(V_2\) which is also a Borel map, and s.t. if \(\tilde{\psi}\) is the associated map of the objects \(U_1\) of \(V_1\) to the \(U_2\) of \(V_2\), then \(\tilde{\psi}^{-1}(A)\) is a \(C_1\)-null set for every saturated analytic \(C_1\)-null set \(A\). A \(\text{homomorphism}\) of \((V_1, C_1)\) to \((V_2, C_2)\) is a functor which is a Borel map, and whose restriction to some inessential contraction of \((V_1, C_1)\) is a strict homomorphism. Two homomorphisms \(\psi_1\) and \(\psi_2\) of \((V_1, C_1)\) to \((V_2, C_2)\) are \(\text{strictly similar}\) if \(\theta(r(f)) \psi_1(f) = \psi_2(f) \theta(d(f))\) for all \(f \in V_1\) and for some Borel map \(\theta: U_1 \rightarrow U_2\) for which both sides are defined; \(\psi_1\) and \(\psi_2\) are \(\text{similar}\) if there is an inessential contraction of \(V_1\) on which they are strictly similar. (\textit{Note.} The composition of two homomorphisms is not necessarily a homomorphism!) Similarity is an equivalence relation, and similarity classes \([\phi]\) of homomorphisms \(\phi\) are preserved under composition. Taking virtual groups as objects and similarity classes of homomorphisms as morphisms, one obtains a category. A locally compact separable group \(G\) is a virtual group when endowed with its Haar measure class. A virtual group \(V\), together with a homomorphism \(\pi: V \rightarrow G\), briefly \(V_{\pi}\), will be called a \(\text{virtual group over} \ G\). The \textit{Mackey}
category $\mathcal{M}(G)$ of $G$ has the virtual groups over $G$ as objects, and as morphisms the similarity classes of homomorphisms which make the obvious triangle over $G$ commute, namely, which commute with the action of $[\pi]$. The category $\mathcal{R}(G)$ of ergodic actions of $G$ has as objects the transformation spaces $T$ of $G$, namely:

(a) $T$ is an analytic Borel space;
(b) the map $(t, x) \mapsto tx$ of $T \times G \to T$ is Borel;
(c) $T$ has a measure class $\mathcal{C}$ which is invariant under the set of Borel automorphisms $t \mapsto tx$;
(d) $(T, \mathcal{C})$ is ergodic: the only invariant Borel sets are null or co-null.

The morphisms of $\mathcal{R}(G)$ are equivalence classes of maps, as follows:

$T_1$ and $T_2$ being objects, consider all Borel maps $\phi: T_1 \to T_2$ s.t.

(a) There is a co-null invariant analytic subset of $T_1$ on which $\phi$ is $G$-equivariant, that is, $\phi(tx) = \phi(t)x$.
(b) If $N$ is a null set in $T_2$, so is $\phi^{-1}(N)$.

Two such maps $\phi$ and $\psi$ are equivalent if there is a Borel map $\alpha: T_1 \to G$ such that $\phi(s)\alpha(s) = \psi(s)$ and $\alpha(sx) = x^{-1}\alpha(s)x$ for all $s$ in some co-null invariant analytic subset of $T_1$. This equivalence is preserved under composition of maps; the equivalence classes are the morphisms of $\mathcal{R}(G)$. For a given ergodic action of $G$ on $T$, give $T \times G$ the product Borel structure and product measure class and define $\pi: T \times G \to G$ as the projection. Defining $(s, x)(t, y) = (s, xy)$ whenever $sx = t$ gives $T \times G$ a groupoid structure whose objects are naturally identical with the set $T$. This construction maps the objects of $\mathcal{R}(G)$ into the objects of the Mackey category $\mathcal{M}(G)$.

**Theorem 1.** There is a functor $R: \mathcal{R}(G) \to \mathcal{M}(G)$ extending the above construction, which is faithful on objects and morphisms and whose image is a full subcategory of the Mackey category.

The easy proof, following Mackey's techniques, is omitted.

**4. Main result.**

**Theorem 2.** The functor $R$ has a left adjoint $M$.

We call $M$ the Mackey functor; a sketch of the proof follows, leaving fine measure-theoretic points to a detailed publication to follow. All homomorphisms and similarities are tacitly taken as strict. Given $V$, we first construct a transformation space $T$ by generalizing a construction of Mackey, which is, in turn, patterned after a classical construction of induced representations. Take $U \times G$, where $U$ is the object of $V$, and define a $G$-action by $(u, y)x = (u, x^{-1}y)$. Set $(u, x)$
~(v, y) iff \((r(f), x) = (u, x)\) and \((d(f), x\pi(f)) = (v, y)\) for some \(f \in V\). The \(G\)-action commutes with the equivalence ~, so that the quotient \(U \times G/\sim\) is a \(G\)-space (but unfortunately, not an analytic Borel \(G\)-space under the quotient Borel structure). It is however possible (using some results of Mackey) to replace \(U \times G/\sim\) by a Borel space \(M(V_\ast)\) having the same measure algebra, and the action of \(G\) is easily seen to be ergodic. This defines the functor \(M\) on the objects; next, we define it on morphisms. Given \([\phi]: V_1, \pi_1 \to V_2, \pi_2\) in \(M(G)\), the map 
\[(u, x) \mapsto (\phi(u), x\theta(u))\] 
of \(U_1 \times G \to U_2 \times G\), where \(\theta(r(f))\pi_2 \circ \phi(f) = \pi_1(f)\theta(d(f))\), factors through ~ to give a map \(U_1 \times G/\sim \to U_2 \times G/\sim\).

If \(\theta\) is changed to another \(\theta_1\) implementing the similarity, then 
\[(\phi(u), x\theta(u))x\theta(u)\theta_1(u)^{-1}x^{-1} = (\phi(u), x\theta_1(u)),\] 
and the map \((u, x) \mapsto x\theta(u)\theta_1(u)^{-1}x^{-1}\) factors to give a map \(U_1 \times G/\sim \to U_2 \times G/\sim\) which belong to the same morphism class in \(R(G)\). Again, if \(\gamma(r(f))\phi(f) = \phi_1(f)\gamma(d(f))\), then \(u \mapsto \theta(u)\pi_2 \circ \gamma(u)^{-1}\) implements the similarity between \(\pi_3 \circ \phi_1\) and \(\pi_1\) (recall \(V_1: \pi \to G\)). The maps 
\[(u, x) \mapsto (\phi(u), x\theta(u))\] 
and 
\[(u, x) \mapsto (\phi(u), x\theta_1(u))\] 
factor to give \(G\)-equivariant maps \(U_1 \times G/\sim \to U_2 \times G/\sim\) which belong to the same morphism class in \(R(G)\). Thus, the map of morphisms in \(M(G)\) to morphisms in \(R(G)\) is well-defined. It can be verified that the definition preserves composition of morphisms. Next, given \([\phi]: V_\ast \to R(T)\), we must associate a morphism \(M(V_\ast) \to T\). Let \(\bar{\phi}\) be the projection of \(\phi\) onto \(T\) and \(\beta\) be the projection of \(T \times G \to G\). For \(\theta(r(f))\beta \circ \phi(f) = \pi_1(f)\theta(d(f))\), the map \((u, x) \mapsto \bar{\phi}(u)\theta(u)^{-1}x^{-1}\) is constant on \(\sim\)-equivalence classes and \(G\)-equivariant, so that it induces a map \(U \times G/\sim \to T\); it is again checked as above that changing \(\theta\) or \(\phi\) does not change the equivalence class of maps. Conversely, to a morphism \([\phi]: M(V_\ast) \to T\) we must associate a morphism \(V_\ast \to R(T)\). Let \(\rho\) be the natural projection of \(U \times G \to U \times G/\sim\), and set \(\tilde{\phi}(f) = (\phi \circ \rho(r(f), e), \pi(f))\), when \(e\) is the identity of \(G\); one can show that \(\tilde{\phi}(fh) = \phi(f)\tilde{\phi}(h)\), that the required diagram commutes, and that changing to an equivalent map changes \(\tilde{\phi}\) only up to similarity. Lastly, one must verify that the two constructions are inverses and naturality of the whole construction. All this can be done with little difficulty.

**BIBLIOGRAPHY**


