PARTIAL DIFFERENTIAL EQUATIONS IN FISCHER-FOCK SPACES FOR THE HILBERT-SCHMIDT HOLOMORPHY TYPE

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1. Introduction. Current work on the extension of function theory to infinite-dimensional domains has led to the consideration of classes of analytic functions defined on Banach spaces, with Fréchet derivatives of a given type, e.g., nuclear, compact or integral. The existence theory of partial differential equations in this setting follows from [G] for the nuclear bounded case, and is given in [D] for formal power series of \( \alpha-\beta-\gamma \)-type. In this note we describe the duality theory (Theorem 1) and the existence theory (Theorem 2) of partial differential equations for a class of spaces of entire functions defined on a Hilbert space, with Fréchet derivatives given by Hilbert-Schmidt operators. When the underlying Hilbert space is finite-dimensional, we recover results in [T, Chapter 9], in [B] and in [NS] (Fischer space). When the underlying space is a Hilbert space of square-integrable functions, we obtain the wave functionals in the Fock representation of quantum field theory (cf. [NT]), subsuming some of the results proved independently in [R].

2. Hilbert-Schmidt polynomials. Let \( E \) be a Hilbert space over the complex field \( \mathbb{C} \), with inner product \( \langle \cdot, \cdot \rangle \), and \( E' \) the dual of \( E \), with the inner product \( \langle u', v' \rangle = \langle v, u \rangle \) for \( u' = (u), v' = (v) \). Let \( E_n \) denote the \( n \)-fold symmetric product of \( E' \) [Gr, p. 191]. The Hilbert-Schmidt inner product on \( E_n \) is characterized for decomposable elements by

\[
\langle u_1 \vee \cdots \vee u_n, v_1 \vee \cdots \vee v_n \rangle = \frac{1}{n!} \sum_{\pi} \langle u_{\pi(1)}', v_1 \rangle \cdots \langle u_{\pi(n)}', v_n \rangle,
\]

the summation extended over all permutations \( \pi \) of the indices. \( E_n \) denotes the \( n \)-fold symmetric product equipped with the Hilbert-Schmidt inner product.

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Schmidt inner product, and \((E_H^n)^\wedge\) its completion.

For \(n = 1, 2, \ldots\), let \(\mathcal{O}(E)\) denote the Banach space of continuous \(n\)-homogeneous polynomials \(P\) (obtained from continuous symmetric \(n\)-linear forms \(A : E \times \cdots \times E \to \mathbb{C}\) by \(P(x) = A(x, \ldots, x)\)), with the supremum norm on the unit ball of \(E\), and let \(\mathcal{O}(E) = \mathbb{C}\) [N, p. 7].

**Proposition 1.** The formula \(i(u_1 \vee \cdots \vee u_n') = u_1' \cdots u_n'\), where \(u_1' \cdots u_n'(x) = u_1'(x) \cdots u_n'(x)\) for \(x \in E\) (also \(u_1' = u' \cdots u'\)), defines a linear injection from \(E_H^n\) into \(\mathcal{O}(E)\). The continuous linear extension \(\hat{i}\) of \(i\) to \(\hat{E}_H^n\) is still injective. The image of \(\hat{i}\), denoted by \(\mathcal{P}(E)\), is the Hilbert space of \(n\)-homogeneous Hilbert-Schmidt polynomials on \(E\), with the inner product inherited from \(E_H^n\) and the associated norm by \(\| \|_H\). Let \(\mathcal{O}(E)^'\) be equipped with the dual inner product. Given \(P_n \in \mathcal{O}(E)^'\), the formula \(P_n'(x') = (x'|P_n)_H\), where \(x \in E\) and \(x'| = (x')\), defines \(P_n' \in \mathcal{O}(E)^'\), and the map \((P_n)_H \mapsto P_n'\) is a Hilbert space isomorphism.

3. Entire functions of Hilbert-Schmidt type.

**Proposition 2.** Given \(\rho > 0\), if \(f_n \in \mathcal{O}(E)\) for each \(n\) and \(\sum_{n=0}^\infty \rho^n \|f_n\|^2_H/n! < \infty\) then \(f = \sum_{n=0}^\infty \rho^n n! f_n\) is an entire function of bounded type, i.e., \(f\) takes bounded sets into bounded sets. If \(d^nf(x)\) denotes the \(n\)th derivative polynomial of \(f\) at \(x\) then \(d^nf(0) = f_n\). The class of such functions, denoted by \(\mathcal{F}_\rho(E)\), is a Hilbert space, with the inner product \((\ , \ )_\rho\) given by

\[
(f, g)_\rho = \sum_{n=0}^\infty \frac{\rho^n}{n!} (d^nf(0), d^n(g(0)))_H
\]

and the associated norm denoted by \(\| \|_\rho\). Clearly \(\| \|_\rho \leq \| \|_\sigma\) when \(\rho \leq \sigma\). Hence \(\mathcal{F}_\rho(E) = \bigcap_{0 < \rho < \infty} \mathcal{F}_\rho(E)\) with the projective limit topology is a countably Hilbert space, thus a reflexive Fréchet space, and \(\mathcal{F}_0(E) = \bigcup_{0 < \rho < \infty} \mathcal{F}_\rho(E)\) with the locally convex inductive limit topology is a bornological \((DF)\)-space.

**Theorem 1.** Let \(0 \leq \rho \leq \infty\), with \(\rho^{-1} = 0\) or \(\infty\) when \(\rho = \infty\) or \(0\). If \(f = \sum_{n=0}^\infty f_n/n! \in \mathcal{F}_\rho(E)\) and \(g' = \sum_{n=0}^\infty g_n'/n! \in \mathcal{F}_{\rho^{-1}}(E')\), and if \(g_n' \in \mathcal{O}(E')\) corresponds to \((g_n)_H \in \mathcal{O}(nE)^'\), then the series

\[
\langle f, g' \rangle = \sum_{n=0}^\infty \frac{1}{n!} (f_n, g_n)_H
\]

defines a bilinear form, placing \(\mathcal{F}_\rho(E)\) and \(\mathcal{F}_{\rho^{-1}}(E')\) in separating duality. The map \(g' \mapsto \langle \ , g' \rangle\) is a Hilbert space isomorphism (resp. a topological vector space isomorphism) from \(\mathcal{F}_\rho(E)^'\) onto \(\mathcal{F}_{\rho^{-1}}(E')\) when
$0 < \rho < \infty$ (resp. $\rho = 0$ or $\infty$), and is the inverse of the Fourier-Borel transformation $[D]$.

**Sketch of Proof.** Let $T \in \mathcal{F}_\rho(E)'$. Since $\mathcal{F}_\rho(E)$ is continuously imbedded in each $\mathcal{F}_\rho(E)$, the restriction of $T$ to $\mathcal{F}_\rho(E)$ belongs to $\mathcal{F}_\rho(E)'$, corresponding to $T'_n \in \mathcal{F}_\rho(E)'$ given by $T'_n(x') = T(x^n)$. The formula $\hat{T}(x') = T(\exp x')$ defines the Fourier-Borel transform $\hat{T} : E' \to \mathbb{C}$ of $T$. We have in fact $\hat{T} = \sum_{n=0}^{\infty} T'_n / n! \in \mathcal{F}_{\rho^{-1}}(E')$. Conversely, given $g' \in \mathcal{F}_{\rho^{-1}}(E')$ each $\check{d}^n g'(0) \in \mathcal{F}_\rho(nE)'$ corresponds to a unique $(|g_n|_H \in \mathcal{F}_\rho(nE)'$. The formula

$$T'_\rho(f) = \sum_{n=0}^{\infty} \frac{1}{n!} \langle \check{d}^n f(0) | g_n \rangle_H$$

for $f \in \mathcal{F}_\rho(E)$ defines $T'_\rho \in \mathcal{F}_\rho(E)'$, and we get $\hat{T}'_\rho = g'$. This establishes the isomorphism of vector spaces and the duality $\langle f, g' \rangle = T'_\rho(f)$. Moreover, $\| g' \|_{\rho^{-1}} = \| T'_\rho \|$ (dual norm in $\mathcal{F}_\rho(E)'$) when $0 < \rho < \infty$. The continuity of the mappings from $\mathcal{F}_\infty(E)'$ (resp. $\mathcal{F}_\rho(E)'$) onto $\mathcal{F}_\rho(E)'$ (resp. $\mathcal{F}_\infty(E)'$) follows from the isometry from $\mathcal{F}_\rho(E)'$ onto $\mathcal{F}_{\rho^{-1}}(E')$ for $0 < \rho < \infty$ by the properties of countably Hilbert spaces, bornological (DF)-spaces and their duals.

**Corollary 1.** $\mathcal{F}_\rho(E)$ is reflexive and complete. $\mathcal{F}_\infty(E)$ and $\mathcal{F}_0(E)$ are Montel spaces, in fact nuclear, if and only if $E$ is finite-dimensional.

**Sketch of Proof.** The reflexivity, hence completeness, of the (DF)-space $\mathcal{F}_\rho(E)$ follows from the duality. In the finite-dimensional case the nuclearity of $\mathcal{F}_\infty(E)$, hence of $\mathcal{F}_0(E)$, comes from the nuclearity of the injections $\mathcal{F}_\rho(E) \to \mathcal{F}_\rho(E)$ for $\rho < \sigma$. $E'$ is a closed barrelled subspace of $\mathcal{F}_\infty(E)$ and of $\mathcal{F}_0(E)$, so these spaces cannot be Montel or nuclear in the infinite-dimensional case.

4. **Partial differential operators of Hilbert-Schmidt type.** To define partial differential operators we need the following inequality.

**Proposition 3.** If $0 \leq k \leq n$ and $P_n \in \mathcal{F}_\rho(nE)$ then $\check{d}^k P_n(x) \in \mathcal{F}_\rho^*(nE)$, and for all $x \in E$ we have:

$$\left\| \frac{1}{k!} \check{d}^k P_n(x) \right\|_H \leq \binom{n}{k} \| P_n \|_H \| x \|^{n-k}.$$

The proof, and others below, makes use of the following representation:

**Lemma 1.** Given an orthonormal basis $(e_i)_i$ of $E$, each $P_n \in \mathcal{F}_\rho(nE)$ is uniquely expressed as a limit in $\| \cdot \|_H$-norm by
\[ P_n = \sum_{i_1, \ldots, i_n} s_{i_1} \cdots s_{i_n} e_{i_1} ' \cdots e_{i_n}' \]

with symmetric coefficients \( s_{i_1} \cdots s_{i_n} \in \mathbb{C} \), and

\[ \| P_n \|^2_H = \sum_{i_1, \ldots, i_n} | s_{i_1} \cdots s_{i_n} |^2. \]

We observe, however, that the \( e_{i_1}' \cdots e_{i_n}' \) are not orthonormal.

By \([N, \S 9, \text{Lemma 1}]\) we get \( d^n f(x) \in \mathcal{D}_H(\mathbb{R}^n) \) for all \( x \in \mathbb{R} \) when \( d^n f(0) \in \mathcal{D}_H(\mathbb{R}^n) \) and \( \limsup_n \| d^n f(0) / n! \|^2_{H^n} = 0 \). We may then define: given \( P = \sum_{n=0}^m P_n \) with \( P_n \in \mathcal{D}_H(\mathbb{R}^n) \), the partial differential operator of Hilbert-Schmidt type \( P(d) \) is given by

\[ P(d)f(x) = \sum_{n=0}^m (d^n f(x) | P_n)_H. \]

If \( P = u_1' \cdots u_n' \) then \( P(d) \) is given by successive directional differentiation along \( u_1, \ldots, u_n \). In particular, we are reduced to linear partial differential operators with constant coefficients in the finite-dimensional case. We also define the multiplication operator \( P \cdot \) by \( P \cdot f(x) = P(x)f(x) \).

**Proposition 4.** If \( f \) is in \( \mathcal{F}_\sigma(\mathbb{R}) \) then \( P(d)f \) and \( P \cdot f \) are in \( \mathcal{F}_\sigma(\mathbb{R}) \) for every \( 0 < \rho < \sigma < \infty \). Hence if \( f \) is in \( \mathcal{F}_\sigma(\mathbb{R}) \) (resp. \( \mathcal{F}_0(\mathbb{R}) \)) then so are \( P(d)f \) and \( P \cdot f \).

Easy counterexamples show that not all \( f \in \mathcal{F}_\sigma(\mathbb{R}) \) are mapped into \( \mathcal{F}_\sigma(\mathbb{R}) \) by \( P(d) \) or \( P \cdot \).

**Theorem 2.** Let \( 0 \leq \rho \leq \infty \) and let \( P(d) \) be any partial differential operator of Hilbert-Schmidt type: then for every \( f \in \mathcal{F}_\rho(\mathbb{R}) \) there is some \( g \in \mathcal{F}_\rho(\mathbb{R}) \) such that \( P(d)g = f \).

The proof uses the following lemmas:

**Lemma 2.** If \( P = \sum_{n=0}^m P_n \) and \( P' = \sum_{n=0}^m P_n' \), where \( ( | P_n)_H \) \( \in \mathcal{D}_H(\mathbb{R}^n) \) corresponds to \( P_n' \in \mathcal{D}_H(\mathbb{R}^n) \) by \( P_n'(x') = (x' | P_n)_H \), then \( \langle P(d)f, g' \rangle = \langle f, P' \cdot g' \rangle \) for \( f \) and \( g' \) in the corresponding dual pairs (Theorem 1), finiteness and equality holding when either side is finite.

The proof follows from a similar identity for the duality between \( \mathcal{D}_H(\mathbb{R}^n) \) and \( \mathcal{D}_H(\mathbb{R}^n) \), established first for polynomials of finite type (i.e., given by \( E^{\nu_n} \)), which are dense in \( \mathcal{D}_H(\mathbb{R}^n) \).

**Lemma 3.** If \( 0 < \rho < \infty \), \( f \in \mathcal{F}_\rho(\mathbb{R}) \) and \( P = \sum_{n=0}^m P_n \) with \( P_n \in \mathcal{D}_H(\mathbb{R}^n) \) then \( \| P \cdot f \| \leq \| P_m \| \| f \| \).

The proof follows from a similar identity for the duality between \( \mathcal{D}_H(\mathbb{R}^n) \) and \( \mathcal{D}_H(\mathbb{R}^n) \), established first for polynomials of finite type (i.e., given by \( E^{\nu_n} \)), which are dense in \( \mathcal{D}_H(\mathbb{R}^n) \).
The proof of the inequality uses a polynomial identity given in [T, Lemma 2.2] applied first to $P$ of finite type and $f \in \mathcal{O}_H(\mathcal{E})$, extended by density to any Hilbert-Schmidt polynomial $P$, and finally to any $f \in \mathcal{H}(\mathcal{E})$, with the help of the following facts: the pairs of operators $e'(d)$ and $e'$ obtained from an orthonormal basis $(e'_i)$ of $\mathcal{E}$ satisfy the correct commutation relations required over the polynomials; and Taylor series converge in $\| \cdot \|_p$-norm. The continuity of $P \cdot f \to f$ follows from the inequality, and from the properties of projective and inductive limits in the cases $p = \infty$ and $p = 0$.

**Proof of Theorem 2.** By Lemma 2 the transpose of $P(d)$ by $(\cdot, \cdot)$ is $P'$, which has a continuous left inverse by Lemma 3 applied to $\mathcal{H}(\mathcal{E})$ (again $p^{-1} = 0$ when $p = \infty$). A standard Hahn-Banach argument gives the result.

**Proposition 5.** Let $M$ be a measure space (e.g., locally compact), and make $\mathcal{E} = L^2(M)$: then $P_n \in \mathcal{O}_H(\mathcal{E})$ if and only if there is some $h_n \in L^2(M \times \cdots \times M)$, $(n$ variables and product measure), such that

$$P_n(\alpha) = \int_M \cdots \int_M h_n(t_1, \cdots, t_n) \alpha(t_1) \cdots \alpha(t_n) \, dt_1 \cdots dt_n$$

for every $\alpha \in L^2(M)$. The function $h_n$ can be uniquely chosen to be symmetric, and then $\|P_n\|_H = \|h_n\|_L^2$.

It follows that the functions $f \in \mathcal{H}(\mathcal{E})$ are the Fock functionals of [NT] and [R], and the partial differentials $P_n(d)f(\alpha)$ are the functional derivatives $h_n f^{(n)}(\alpha)$ of [R], where $h_n$ corresponds to $P_n$ by the formula given above. The proof of Proposition 5 follows from the Hilbert space isomorphism between $(L^2(M)_H^{(n)})^\wedge$ and symmetric $L^2(M \times \cdots \times M)$.

**References**


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