ON THE CONVERGENCE OF MULTIPLE FOURIER SERIES

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We continue from [2].

THEOREM. Let $P$ be an open polygonal region in $\mathbb{R}^2$, containing the origin. Set $\lambda P = \{(\lambda x, \lambda y) | (x, y) \in P\}$ for $\lambda > 0$. Then for

$$f \sim \sum_{m,n=-\infty}^{\infty} a_{mn} \exp[i(mx + ny)]$$

in $L^2([0, 2\pi] \times [0, 2\pi])$, we have

$$f(x, y) = \lim_{\lambda \to \infty} \sum_{(m,n) \in \lambda P} a_{mn} \exp[i(mx + ny)]$$

almost everywhere.

Surprisingly, this is an easy consequence of Carleson’s theorem [1] on convergence of Fourier series of one variable.

PROOF. It is enough to prove the maximal inequality

(1) $$\left\| \sup_{\lambda} \sum_{(m,n) \in \lambda P} a_{mn} \exp[i(mx + ny)] \right\|_2 \leq C \|f\|_2.$$ 

Inequality (1) follows from the special case in which $P$ is a triangle with a vertex at the origin; for any polygon breaks up into triangles, and the characteristic function of any triangle is a linear combination of characteristic functions of triangles with vertices at zero. Consequently, we can assume $P$ has the form $P = \{(x, y) \in S | (x, y) \cdot t < a\}$, where $S$ is a sector of angle $< \pi$ emanating from the origin, $t \in \mathbb{R}^2$, and $a \in \mathbb{R}^1$. Thus (1) is equivalent to

(2) $$\left\| \sup_{\theta \in \mathbb{R}^2} \sum_{(m,n) \in S; (m,n) \cdot t < \theta} a_{mn} \exp[i(mx + ny)] \right\|_2 \leq C \|f\|_2.$$ 

Evidently, it suffices to prove (2) for rational $t$ (with $C$ independent of $t$), and to do so it is clearly enough to deal with the case $t = (p, q)$ where $p$ and $q$ are relatively prime integers. Finding integers $r$ and $s$ for which $pr - qs = 1$, we let the matrix $A = \left( \begin{smallmatrix} p & q \\ q & -p \end{smallmatrix} \right) \in SL(2, \mathbb{Z})$ act as an automorphism of the 2-torus. Under the action of $A$, (2) becomes

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Here,

\[ \sup_{b} \left| \sum_{(m',n') \in B':m' < b} a_{m'\cdot n'} \exp\left[i(m'x' + n'y')\right] \right|_2 \leq C \|f'\|_2. \]

is the Fourier series of \( f' \). Note that \( C \) is unchanged from (2) to (3). However, (3) follows at once by applying the Carleson-Hunt theorem of [3] to the function \( g(\cdot, y') \) for each \( y' \), where \( g'(x', y') \sim \sum_{(m',n') \in B'} a_{m'\cdot n'} \exp[i(m'x' + n'y')] \). Q.E.D.

REMARKS. 1. The same proof applies to all \( L^p \), \( p > 1 \), and also (with some padding) to polyhedra in \( n \) variables.

2. For \( P \) a rectangle, a more precise argument, discovered independently by P. Sjölin [4], proves convergence of double Fourier series under minimal growth conditions on \( f \). The best known hypotheses are \( f \in L^2(\log L)^2 \log \log L \) for \( P \) a rectangle, and \( f \in L^2(\log L)^3 \log \log L \) in general. The relationship of our proof to Sjölin’s is not clear.

3. N. Tevzadze [5] has shown that for \( f \in L^2([0, 2\pi] \times [0, 2\pi]) \) and for any monotone sequence of rectangles \( R_1 \subseteq R_2 \subseteq R_3 \subseteq \cdots \) in \( R^3 \) with sides parallel to the coordinate axes,

\[ f(x, y) = \lim_{i \to \infty} \sum_{(m,n) \in R_i} a_{mn} \exp[i(mx + ny)] \]

almost everywhere.

Compare with the counterexamples of [2].

REFERENCES


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