GEOMETRIC THEORY OF DIFFERENTIAL EQUATIONS.
THE LJAPUNOV INTEGRAL FOR
MONOTONE COEFFICIENTS

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Communicated by James Serrin, March 3, 1971

An equation
\[ x'' + p(t)x = 0, \quad -\infty < t < \infty, \quad p(t) > 0, \]
can be considered as the Frenet equation of a locally convex curve
\[ x(t) = (x_1(t), x_2(t)) \]
in unimodular centroaffine differential geometry [1]. The osculating ellipse \( E_u \) at \( x(u) \) is the solution of
\[ y'' + p(u)y = 0, \quad y(u) = x(u), \quad y'(u) = x'(u). \]

We prove an unimodular centroaffine Kneser theorem:

**Theorem 1.** If \( p(t) \) is strictly monotone and differentiable in an interval \([a, b]\), then every osculating ellipse \( E_t \), \( t \in [a, b] \), contains all osculating ellipses of smaller area defined on the same interval in its interior.

The area of the osculating ellipse is proportional to \( p(t)^{-1/2} \). By the Jordan curve theorem, the assertion is true if it is true for neighboring points. Then it is easily checked that a pair of conjugate diameters of the smaller ellipse is in the interior of the larger one. The approximation and convergence theorems of convexity imply:

**Theorem 2.** If \( p(t) \) is monotone and continuous in \([a, b]\), then every osculating ellipse \( E_t \), \( t \in [a, b] \), contains all osculating ellipses of smaller area defined on the same interval.

The parameter \( t - u \) is equal to two times the area covered by the radius vector of \( x(t) \) and \( \int_u^t p(\tau) d\tau \) is equal to two times the area covered by the radius vector of the polar reciprocal \( x^*(t) \) of \( x \) for the unit circle, if \( x(t) \) is a curve of unit Wronskian [1, §3]. For \( p \) monotone increasing, the curve \( x \) and the osculating ellipses \( E_t (t \geq u) \) are contained in \( E_u \), and \( x^*(\tau), E^*_t (u \leq \tau \leq t) \) are in \( E^*_t \).

Let \( \phi(u) \) be the conjugate point of \( u \) for (1), i.e., the zero following

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AMS 1970 subject classifications. Primary 34B05, 53A40, 34C10; Secondary 34A40, 53A15.

Key words and phrases. Unimodular centroaffine differential geometry, Kneser theorem, Lyapunov integral, conjugate point, co-conjugate point.

Research partially supported by NSF Grant No. GP-19133.
$u$ of a nontrivial solution of (1) that vanishes at $u$. Let $\psi(u)$ be the co-conjugate point of $u$ for (1), i.e., the zero following $u$ of the derivative of a nontrivial solution of (1) whose derivative vanishes at $u$. A major topic in the study of equations (1) are estimates of the Ljapunov integral

$$L(u) = [\phi(u) - u] \int_u^{\phi(u)} \rho(t) \, dt.$$  

We suppose that $\rho(t)$ is monotone increasing and continuous and that $\phi(u) < \infty$. As an application of Theorem 2, we have

$$\pi \rho(\phi(u))^{-1/2} \leq \phi(u) - u \leq \pi \rho(\phi(u))^{-1/2},$$

$$\pi \rho(u)^{1/2} \leq \int_u^{\phi(u)} \rho(t) \, dt.$$  

If $\psi(u) \geq \phi(u)$, then

$$\int_u^{\phi(u)} \rho(t) \, dt \leq \pi \rho(\phi(u))^{1/2}.$$  

If $\psi(u) < \phi(u)$, then

$$\int_u^{\phi(u)} \rho(t) \, dt \leq \frac{3}{2} \pi \rho(\phi(u))^{1/2}.$$  

(The difference $\phi(u) - \psi(u)$ has been investigated in [2].) Together, we obtain:

**Theorem 3.** For monotone increasing, continuous, positive $\rho(t)$, the Ljapunov integral satisfies

$$\pi^2 \left( \frac{\rho(u)}{\rho(\phi(u))} \right)^{1/2} \leq L(u) \leq \left[ 1 + \frac{1 + \epsilon}{4} \right] \pi^2 \left( \frac{\rho(\phi(u))}{\rho(u)} \right)^{1/2},$$

$$\epsilon = \text{sgn} [\phi(u) - \psi(u)].$$

**References**


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