INTEGRABILITY CONDITIONS FOR $\Delta u = k - Ke^{au}$ WITH APPLICATIONS TO RIEMANNIAN GEOMETRY

BY JERRY L. KAZDAN and F. W. WARNER

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1. In this note we announce some integrability conditions for the equation $\Delta u = k - Ke^{au}$ on compact orientable Riemannian 2-manifolds (where $\Delta$ is the Laplacian), and we give some applications to problems in Riemannian geometry. Further results and details will appear elsewhere. We begin with a description of the geometry problem which led us to a study of the above equation. $M$, throughout, will denote a compact, connected, oriented, 2-dimensional manifold.

Problem. What are necessary and sufficient conditions on a sufficiently smooth (we shall restrict ourselves to $C^\infty$ data here) function $K$ on $M$ for $K$ to be the Gaussian curvature of some Riemannian metric on $M$?

If $K$ is the Gaussian curvature of a Riemannian metric $g$ with volume form $\omega$ on $M$, the only known global condition which $K$ must satisfy is the Gauss-Bonnet formula

$$\int_M K\omega = 2\pi \chi(M),$$

where $\chi(M)$ is the Euler characteristic of $M$. Here $K\omega$ is called the curvature form of the metric $g$. One can rephrase the above question in terms of curvature forms, and in this case it turns out [9] that the condition $\int_M \Omega = 2\pi \chi(M)$ is not only necessary but also is sufficient for a two form $\Omega$ to be the curvature form of a Riemannian metric on $M$. As for the Gaussian curvature functions themselves, (1) seems only to impose certain sign requirements depending on the genus of $M$. Specifically, it seems natural to expect that a necessary and sufficient condition for a smooth function $K$ to be the Gaussian curvature of a Riemannian metric on $M$ is

(i) that $K$ be positive somewhere if genus$(M) = 0$,
(ii) that $K$ change sign, if not identically 0, if genus$(M) = 1$,
(iii) that $K$ be negative somewhere if genus$(M) > 1$.

As a special case of (i), H. Gluck has recently shown [3] that $K$ is a Gaussian curvature if $K$ is strictly positive. His approach is to

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show that by composing \( K \) with a diffeomorphism one can satisfy the integrability conditions for the Minkowski problem. This approach, however, is limited to the case of genus 0 and strictly positive \( K \).

Let \( g \) be a given Riemannian metric on \( M \) with Gaussian curvature \( k \). We attack the above problem by trying to realize \( K \) (or \( K \circ \phi \) where \( \phi \) is an arbitrary diffeomorphism of \( M \)) as the curvature of a metric \( \bar{g} \) pointwise conformal to \( g \), that is, of the form \( \bar{g} = e^{2u}g \) for some \( C^\infty \) function \( u \) on \( M \). This approach has the advantage that, in principle at least, it is applicable to all cases (i)–(iii) and it leads directly to the specific partial differential equation for \( u \),

\[
\Delta u = k - Ke^{2u},
\]

where \( \Delta \) is the Laplacian of the metric \( g \).

In this form our problem is related to a question posed by L. Nirenberg who asked, for a given smooth strictly positive function \( K \) on \( S^2 \), whether or not there is a compact strictly convex surface \( \Sigma \) in \( E^3 \) and a conformal diffeomorphism \( \phi: \Sigma \to S^2 \) such that \( K \circ \phi \) is the Gaussian curvature of \( \Sigma \). This reduces directly to the question of the existence of solutions of \( \Delta u = 1 - Ke^{2u} \) on \( S^2 \) for a given strictly positive \( K \). It has been shown by D. Koutroufiotis [6] that this equation does have solutions for symmetric functions \( K \) on \( S^2 \) sufficiently close to the function 1. However, it is a consequence of one of our integrability conditions (see §3 below) that there are \( K \) arbitrarily close to 1 for which this equation has no solutions.

Melvyn Berger pointed out to us some work [1] that he had done on equation (2) by variational methods, and using these methods he has made some progress [2] on our questions (ii) and (iii), answering (iii) affirmatively for strictly negative \( K \) and providing a partial solution for (ii), a complete solution for which we announce below.

2. In this section \( M \) is a compact, connected, oriented, Riemannian 2-manifold, with \( \Delta \) the associated Laplace operator and \( \omega \) the volume form. Let \( f \) be a \( C^\infty \) function on \( M \) with \( \int_M f\omega = 0 \). In this situation we have a necessary and sufficient condition on \( h \) for there to exist a solution of \( \Delta u = f + h e^{au} \) for \( \alpha \) a positive real constant.

**Theorem 1.** We consider the equation \( \Delta u = f + h e^{au} \) under the assumptions that \( \int_M f\omega = 0 \) and \( \alpha > 0 \). If \( h \equiv 0 \), the equation has a solution. If \( h \) is not identically zero, then a necessary and sufficient condition for there to exist a solution is that \( h \) take on both positive and negative values, and that \( \int_M h e^{au}\omega > 0 \), where \( v \) is a solution of \( \Delta v = f \).

The proof uses a variational argument together with an extension
of the Trudinger Inequality [8], [7], [5] to manifolds. In the case of the torus, this gives necessary and sufficient conditions for curvature functions to be related by a conformal change of metric. As an application of Theorem 1 we prove

**Theorem 2.** A necessary and sufficient condition for a smooth function $K$ to be the Gaussian curvature of some Riemannian metric on the torus is that $K$ change sign if not identically 0.

A more subtle consequence of Theorem 1 is the following result, which one might expect since there is no Gauss-Bonnet theorem for the plane $E^2$.

**Theorem 3.** Each $C^\infty$ function on the plane $E^2$ is the Gaussian curvature of some Riemannian on $E^2$.

3. In this section we consider the equation $\Delta u = 1 - Ke^{2u}$ on the 2-sphere $S^2$, where $\Delta$ is the Laplacian of the standard metric. Our main result is the following integrability condition, which shows, among other things, that there are functions $K$ on $S^2$ which are known to be curvature functions but which cannot be realized by a conformal change of the standard metric.

**Theorem 4.** A necessary condition on $K$ for there to exist a solution of $\Delta u = 1 - Ke^{2u}$ on $S^2$ is that

$$\int_{S^2} (e^{2u}\nabla K \cdot \nabla F)\omega = 0$$

for all spherical harmonics $F$ of degree 1. (Here $\nabla$ denotes the gradient on $S^2$.)

This necessary condition can easily be generalized to cover the equation $\Delta u = k - Ke^{2u}$ with $\int_{S^2} k\omega = 4\pi$.

It follows immediately, for example, that $\Delta u = 1 - Ke^{2u}$ has no solutions if $K$ is a spherical harmonic of degree 1 since in this case the integral in (3) is necessarily positive for $K = F$. Since the integral in (3) is unchanged by adding constants to $K$, one can easily construct examples of strictly positive $C^\infty$ functions on $S^2$, for example $2 + \cos \phi$ (we use spherical coordinates $z = \cos \phi$, $x = \sin \phi \cos \theta$, $y = \sin \phi \sin \theta$) for which the equation has no solutions, thereby answering Nirenberg's question negatively. If one takes the special case in which $F = \cos \phi$, then (3), in spherical coordinates, becomes

$$\int_0^\pi \left\{ \int_0^{2\pi} e^{2u}K\phi \sin^2 \phi \ d\theta \right\} \ d\phi = 0.$$
It was this form of (3) to which we were first led by our observation of the nonexistence of rotationally symmetric (function of \( \phi \) alone) solutions for \( \Delta u = 1 - K e^{2u} \) given rotationally symmetric data \( K \) (see [4]).

It appears that (3) poses no a priori constraint if one allows the modification of \( K \) by a diffeomorphism \( \phi \) of \( S^2 \). Thus it is possible that for each \( K \) which is positive somewhere on \( S^2 \) there is a diffeomorphism \( \phi \) of \( S^2 \) such that \( \Delta u = 1 - (K \circ \phi) e^{2u} \) has a solution. If this be the case, then \( K \circ \phi \) and hence \( K \) would be Gaussian curvatures of Riemannian metrics on \( S^2 \), thereby answering (i) affirmatively.

4. The equation in §3 becomes much more interesting if we free it from the geometric case of exponent 2 in \( e^{2u} \) and consider the equation \( \Delta u = f + h e^{au} \) on an arbitrary compact, connected, oriented Riemannian 2-manifold \( M \), under the assumptions that \( \int_M f \omega > 0 \) and that \( h \) be negative somewhere. In this situation we have the following theorem, which has been observed in a special case by Berger and Moser.

**Theorem 5.** Suppose that \( (\text{vol}(M))^{-1} \int_M f \omega = c > 0 \), that \( h \) is negative somewhere on \( M \), and that \( \alpha \) is a positive real constant. Then there exists \( \beta > 0 \) such that for \( 0 < c \alpha < \beta \), the equation \( \Delta u = f + h e^{au} \) always has a solution.

As in the case of Theorem 1 the proof here uses a variational argument together with the Trudinger Inequality on manifolds. Using his sharp version of the Trudinger inequality for \( S^2 \) [7], Moser has shown that one can take \( \beta = 2 \) on \( S^2 \). Our Theorem 4 shows that 2 is the best possible value for \( \beta \) on \( S^2 \). One of the interesting phenomena here is the different nature of the constraints at the extremes of the range \( 0 < c \alpha < \beta \). At the value \( c = 0 \), corresponding to our Theorem 1, where we have a necessary and sufficient condition, the constraint is an integral inequality on \( h \) plus the requirement that \( h \) take on both positive and negative values. On \( S^2 \), for \( c \alpha = 2 \), the only known constraint so far is the “Gauss-Bonnet theorem” and an integral identity involving the derivatives of \( h \). The fact that 2 is the best possible value of \( \beta \) for \( S^2 \) is intimately tied to the fact that 2 is the lowest nonzero eigenvalue of \( (-\Delta) \). We have no information on \( S^2 \) for the range \( 2 < c \alpha < 6 \). But again at 6, which is the next eigenvalue of \( (-\Delta) \) we have a constraint analogous to (3) showing that there are rotationally symmetric \( K \), for example \( K = 3 \cos^2 \phi - 1 \), for which \( \Delta u = 1 - Ke^{6u} \) has no rotationally symmetric solutions.

**ADDED IN PROOF.** It follows from a strengthened version of The-
oorem 1 that a necessary and sufficient condition for a smooth function $K$ on the Klein bottle to be the Gaussian curvature of some metric is that $K$ change sign if not identically zero.

Recently, Moser has shown that (2) has antipodally symmetric solutions on $S^2$ if $k=1$ and if $K$ is an antipodally symmetric function which is positive somewhere. From this it follows that the condition of being positive somewhere characterizes curvature functions on the real projective plane.

**References**


University of Pennsylvania, Philadelphia, Pennsylvania 19104