RESEARCH ANNOUNCEMENTS

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CAUSALLY ORIENTED MANIFOLDS AND GROUPS

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A $C^\infty$ manifold is said to be causally oriented if there is given in the tangent plane at each point $p$ a nontrivial convex cone defined locally by $C^\infty$ inequalities. A time-like arc is an oriented $C^\infty$ curve whose forward tangent at each point lies in $C(p)$; the manifold is strongly causal if no nontrivial time-like arc is closed. There is then a partial ordering $x < y$ on $M$, defined by the existence of a nontrivial time-like arc with initial point $x$ and terminal point $y$. If neither $x < y$ nor $y < x$, $x$ and $y$ are incommunicable; a space-like submanifold is a submanifold, any two of whose points are incommunicable. These notions are in part abstractions of some of those treated in [1].

A temporal displacement $T$ is an automorphism of $(M, C)$ such that either $x < Tx$ for all $x \in M$ ("forward displacement"), or $Tx < x$ for all $x \in M$, or $Tx = x$ for all $x \in M$. A causally oriented manifold $(M, C)$ is said to be homogeneous if there exists a maximal space-like surface $S$, on which the subgroup of automorphisms leaving $S$ fixed as a set is transitive, both on the points of $S$ and on the directions at each point and a smooth one-parameter group $T_t$ of temporal displacements such that $M = \bigcup_{t \in \mathbb{R}} T_t(S)$.

**Theorem.** The finite coverings of the conformal compactification $\bar{M}$ of $n$-dimensional Minkowski space-time $M$ admit causal orientations compatible with that in Minkowski space, but are not strongly causal.


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However, the universal covering space $\tilde{M}$ of $M$ is strongly causal, and homogeneous with $S = S^{n-1}$.

The conformal automorphism group $\Gamma (\cong O(n, 2))$ of $\tilde{M}$ is transitive, as a result of which $\tilde{M}$, and hence also any covering space of $\tilde{M}$, is conformally locally identical to $M$. Since Maxwell's and similar equations (wave, Dirac with zero mass, etc.) are uniquely determined by the conformal structure, it follows from [1] that

**Corollary 1.** On the universal covering space of the conformal compactification of Minkowski space-time, Maxwell's equations (etc.) admit global retarded and advanced elementary solutions.

Colloquially speaking, this means that these equations are "quantizable" on $\tilde{M}$, but not on finite coverings of $\tilde{M}$.

A topological group $G$ is said to be causally oriented if there is given in it a nontrivial subset $C$ with the properties: $C^a \subset C$, $a^{-1}Ca \subset C$ if $a$ is in the component of $e$ in $G$, and $C \cap C^{-1} = \{e\}$. In general, an open simple Lie group admits no causal orientation, but

**Corollary 2.** The universal covering group of the conformal group $\Gamma$ is causally oriented by the designation of $C$ as the set of all forward displacements, together with the identity, in its action as a group of conformal automorphisms of $M$.

**Remark 1.** There are no presently known strongly causal homogeneous 4-manifolds other than the two involved in classical and relativistic mechanics, and $\tilde{M}$ (in the case $n = 4$).

**Remark 2.** The infinitesimal generator of the temporal development group involved in the homogeneity of $\tilde{M}$ has been studied in another connection in [3], and shown to have a discrete spectrum in certain unitary representations of $\Gamma$. It follows that it is not conjugate to the standard relativistic generator; it is however generic in a sense in which the standard generator appears as a singular special case.

**References**


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