AN EXACT SEQUENCE INVOLVING THE
CHERN CHARACTER

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In [3] the author defined maps \( b'_n : U(n) \rightarrow \Omega^2 U(n+1) \) which are deformations of the classical Bott homotopy equivalence \( b : U \rightarrow \Omega^2 U \) [1], i.e., the composite \( U(n) \rightarrow \Omega^2 U(n+1) \rightarrow \Omega^2 U \) is homotopic to the composite \( U(n) \rightarrow U \rightarrow \Omega^2 U \). The maps \( b'_n \) are natural with respect to the inclusions \( U(k) \subset U(n) \) for \( k \leq n \). The maps \( b'_n \) may be used to define homomorphisms \( B_n : \pi_r(U(n)) \rightarrow \pi_{r+2}(U(n+1)) \) as the composite homomorphism

\[
\pi_r(U(n)) \xrightarrow{b'_n} \pi_r(\Omega^2 U(n+1)) \xrightarrow{\partial^{-2}} \pi_{r+2}(U(n+1)).
\]

The advantage gained by using the maps \( B_n \) is that they give information on the nonstable homotopy of \( U(n) \) not available from the classical Bott maps, and they agree with the classical results in the stable range. For example, the results of [3] show that the map \( B_n : \pi_r(U(n)) \rightarrow \pi_{r+2}(U(n+1)) \) is an isomorphism for \( r \leq 2n-1 \), and \( B_n : \pi_{2n}(U(n)) \rightarrow \pi_{2(n+1)}(U(n+1)) \) is a monomorphism. Kenneth Millett has calculated \( B_n : \pi_{2(n+r)}(U(n)) \rightarrow \pi_{2(n+r+1)}(U(n+1)) \) for \( r = 2, 3 \).

The purpose of this announcement is to describe an application of the maps \( b'_n \) to complex \( K \)-theory. We work throughout in the category of finite CW complexes with basepoint. We use \( Q \) to denote the additive group of rational numbers, and \( Z \) to denote the group of integers.

1. The spectrum \( TU \). We use the maps \( b'_n \) to define a spectrum \( TU \) by setting \( TU_{2k} = \Omega U(k) \), \( TU_{2k+1} = U(k) \) for \( k \geq 0 \), and \( TU_m = \text{point} \) for \( m < 0 \). The maps of the spectrum are

\[
\tau_{2k} = \text{id} : TU_{2k} = \Omega U(k) \rightarrow \Omega U(k) = \Omega TU_{2k+1}
\]

and

\[
\tau_{2k+1} = b'_k : TU_{2k+1} = U(k) \rightarrow \Omega^2 U(k+1) = \Omega TU_{2k+2}.
\]

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We call\( TU \) the nonstable unitary spectrum.

To compute the homotopy of this spectrum, we use the maps\( B_n : \pi_r(U(n)) \to \pi_{r+2}(U(n+1)) \) mentioned above and results about the iterates of these maps.

**Theorem 1.** The homotopy groups of\( TU \) are as follows
\[
\begin{align*}
\pi_{2s-1}(TU) & \cong \mathbb{Q}/\mathbb{Z} \quad \text{for } s \geq 0, \\
\pi_{2s}(TU) & \cong \mathbb{Z} \quad \text{for } s < 0, \\
\pi_r(TU) & = 0 \quad \text{otherwise.}
\end{align*}
\]

2. **Relative spectra and cohomology.** A relative spectrum\( (E, F) \) consists of spectra\( E \) and\( F \) and inclusion maps\( F_k \subset E_k \) such that the following diagram is homotopy commutative
\[
\begin{array}{cccc}
F_k & \xrightarrow{\epsilon_k'} & F_{k+1} \\
\cap & \searrow & \cap \\
E_k & \xrightarrow{\epsilon_k} & E_{k+1}.
\end{array}
\]

By using the co-exact sequence of pairs
\[
(X, *) \to (X, X) \to (CX, X) \to (SX, *) \to (SX, SX) \to \cdots
\]
and taking direct limits of the exact sequences of homotopy sets
\[
\cdots \to [S^{k+1-n}X; E_k] \to [CS^{k-n}X, S^{k-n}X; E_k, F_k] \to [S^{k-n}X; F_k] \to [S^{k-n}X; E_k] \to \cdots
\]
where\( (E, F) \) is a relative spectrum, one obtains an exact cohomology sequence similar to the ordinary cohomology coefficient sequence.

**Theorem 2.** If\( (E, F) \) is a relative spectrum, there is a long exact sequence
\[
\cdots \to h^{n-1}(X; E) \xrightarrow{j_*^*} h^{n-1}(X; E, F) \xrightarrow{\beta} h^n(X; F) \xrightarrow{i_*^*} h^n(X; E) \to \cdots.
\]

The unitary spectrum\( BU \) is defined by\( BU_{2k} = \Omega U \) and\( BU_{2k+1} = U \) with maps\( \text{id}: BU_{2k} = \Omega U \to \Omega U = \Omega BU_{2k+1} \) and\( b: BU_{2k+1} = U \to \Omega^2 U = \Omega BU_{2k+2}, \) where\( b \) is the classical Bott homotopy equivalence. We easily check that\( (BU, TU) \) is a relative spectrum and thus obtain

**Corollary 3.** There is a long exact cohomology sequence
\[
\cdots \to h^{n-1}(X; BU) \xrightarrow{j_*^*} h^{n-1}(X; BU, TU) \xrightarrow{\beta} h^n(X; TU) \xrightarrow{i_*^*} h^n(X; BU) \to \cdots.
\]

Homotopy groups of a relative spectrum \((E, F)\) are defined in the usual way, and there is a long exact sequence of homotopy groups of spectra involving the homotopy groups \(\pi_n(E, F)\). For the relative spectrum \((BU, TU)\), a calculation shows that the following result holds.

**Theorem 4.** The homotopy groups of \((BU, TU)\) are as follows
\[
\pi_{2s}(BU, TU) \cong \mathbb{Q} \quad \text{for } s \geq 0,
\]
\[
\pi_s(BU, TU) = 0 \quad \text{otherwise.}
\]
Moreover, the exact sequences \(0 \to \pi_{2s}(BU) \to \pi_{2s}(BU, TU) \to \pi_{2s-1}(TU) \to 0\) for \(s \geq 0\) and \(0 \to \pi_{2s}(TU) \to \pi_{2s}(BU) \to 0\) for \(s < 0\) are just the exact sequences \(0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Z} / \mathbb{Q} \to 0\) and \(0 \to \mathbb{Z} \to \mathbb{Z} \to 0\), respectively. □

3. Interpretation of the exact cohomology sequence. In the cohomology sequence, the terms \(h^n(X; BU) \cong \tilde{K}^n(X)\), reduced complex K-theory. (Recall we are working in the category of based complexes.)

The groups \(h^n(X; BU, TU)\) and the maps \(j_*: h^n(X; BU) \to h^n(X; BU, TU)\) are determined by the following proposition. Let \(H^*\) denote ordinary singular cohomology.

**Proposition 5.** For each \(n\), \(h^n(X; BU, TU) \cong \sum_{r \geq 0} \tilde{H}^{n+2r}(X; \mathbb{Q})\). The map \(j_*\) is the truncated (from below) Chern character
\[
j_* = ch' = \sum_{r \geq 0} ch_{n+2r}: \tilde{K}^n(X) \to \sum_{r \geq 0} \tilde{H}^{n+2r}(X; \mathbb{Q}). \quad □
\]

Note that for \(n \leq 1\), \(ch' = ch\), the Chern character. This proposition is proved using theorems about generalized cohomology theories derived from the spectral sequence. See Dyer [2, Chapter 1].

Let \(T^n(X) = h^n(X; TU)\). It remains to analyze \(T^n(X)\) and the maps \(i_*: T^n(X) \to \tilde{K}^n(X), \beta: h^n(X; BU, TU) \to T^{n+1}(X)\). An easy calculation with the Chern character and the truncated Chern character establishes

**Proposition 6.** For each \(n\), \(T^{n+1}(X) = \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes \mathbb{Q} / \mathbb{Z} + T^{n+1}(X)\). The map \(\beta\) has cokernel isomorphic to \(\sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes \mathbb{Q} / \mathbb{Z}\). The map \(i_*\) is a monomorphism \(\tilde{T}^{n+1}(X) \to \tilde{K}^{n+1}(X)\). □

**Remarks.** (i) Although \(\text{Im } \beta \cong \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes \mathbb{Q} / \mathbb{Z}\), the map \(\beta\) is not \(1 \otimes \rho: \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes \mathbb{Q} \to \sum_{r \geq 0} \tilde{H}^{n+2r}(X) \otimes \mathbb{Q} / \mathbb{Z}\) where \(\rho\) is the projection \(\mathbb{Q} \to \mathbb{Q} / \mathbb{Z}\).

(ii) The direct sum splitting of \(T^n(X)\) is not natural with respect to maps \(f: X \to Y\), as can be seen by using the inclusion map \(RP^{2n-1} \to RP^{2n}\).
(iii) The long exact cohomology sequence decomposes into exact sequences of length four

\[ 0 \to \tilde{\mathcal{T}}^n(X) \to \bar{\mathcal{K}}^n(X) \xrightarrow{ch'} \sum_{r \geq 0} \mathcal{H}^{n+2r}(X; \mathbb{Q}) \to \sum_{r \geq 0} \mathcal{H}^{n+2r}(X) \otimes \mathbb{Q}/\mathbb{Z} \to 0 \]

although this is not a natural decomposition.

Let \( \text{tors}(G) \) denote the torsion subgroup of \( G \). An analysis of \( \mathcal{T}^n(X) \) yields

**Proposition 7.** (i) \( \text{For } n \leq 1, \mathcal{T}^n(X) \cong \text{tors}(\bar{\mathcal{K}}^n(X)). \)

(ii) \( \text{For } 1 < n \leq \dim X, \)

\[ \mathcal{T}^n(X) \cong \text{tors}(\bar{\mathcal{K}}^n(X)) \oplus \sum_{r < 0} \mathcal{H}^{n+2r}(X) / \text{tors}(\mathcal{H}^{n+2r}(X)). \]

(iii) \( \text{For } \dim X < n, \mathcal{T}^n(X) \cong \bar{\mathcal{K}}^n(X). \)

In this proposition, \( \dim X \) may be interpreted as the rational singular cohomological dimension.

The preceding propositions are collected in the following

**Theorem 8.** For each finite CW complex \( X \), there is a long exact sequence \(( -\infty < n < \infty )\),

\[ \cdots \to \sum_{r \geq 0} \mathcal{H}^{n-1+2r}(X) \otimes \mathbb{Q}/\mathbb{Z} \oplus \mathcal{T}^n(X) \xrightarrow{\text{ch}'} \bar{\mathcal{K}}^n(X) \to \sum_{r \geq 0} \mathcal{H}^{n+2r}(X; \mathbb{Q}) \]

\[ \to \sum_{r \geq 0} \mathcal{H}^{n+2r}(X) \otimes \mathbb{Q}/\mathbb{Z} \oplus \mathcal{T}^{n+1}(X) \to \bar{\mathcal{K}}^{n+1}(X) \to \cdots. \]

Detailed proofs, applications, and extensions of these results will appear elsewhere.

**Bibliography**


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