WEIGHTED APPROXIMATION OF CONTINUOUS FUNCTIONS

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1. Notation. Let $X$ be a completely regular Hausdorff space and $E$ a (real or complex) locally convex Hausdorff space. $F(X, E)$ is the vector space of all mappings from $X$ into $E$, and $C(X, E)$ is the vector subspace of all such mappings that are continuous. $B_\infty(X, E)$ is the vector subspace of $F(X, E)$ consisting of those bounded $f$ that vanish at infinity. The vector subspace $C(X, E) \cap B_\infty(X, E)$ is denoted by $C_\infty(X, E)$. If $X$ is locally compact, $\mathfrak{X}(X, E)$ will denote the subspace of $C(X, E)$ consisting of those functions that have compact support. The corresponding spaces for $E = \mathbb{R}$ or $\mathbb{C}$ are written omitting $E$.

A weight $v$ on $X$ is a nonnegative upper semicontinuous function on $X$. A directed family of weights on $X$ is a set of weights on $X$ such that given $u, v \in V$ and $\lambda \geq 0$ there is a $w \in W$ such that $\lambda u, \lambda v \leq w$. If $U$ and $V$ are two directed families of weights on $X$ and for every $u \in U$ there is a $v \in V$ such that $u \leq v$, we write $U \leq V$. If $V$ is a directed family of weights on $X$, the vector space of all $f \in F(X, E)$ such that $vf \in B_\infty(X, E)$, for any $v \in V$, is denoted by $FV_\infty(X, E)$ and is called a weighted function space. On $FV_\infty(X, E)$ we shall consider the topology determined by all the seminorms $\|f\| = \sup \{v(x) p(f(x)) ; x \in X\}$ where $v \in V$ and $p$ is a continuous seminorm on $E$. $CV_\infty(X, E)$ will denote the subspace $FV_\infty(X, E) \cap C(X, E)$, equipped with the induced topology. The weighted function spaces $CV_\infty(X, E)$ will be called Nachbin spaces.

2. Completeness properties of Nachbin spaces [6]. If for every $x \in X$ there is a weight $u \in U$ such that $u(x) > 0$, we write $U > 0$.

**Lemma.** If $E$ is complete and $U > 0$, then $FV_\infty(X, E)$ is complete.

**Theorem 1.** Suppose that $E$ is complete, and $U$ and $V$ are two directed families of weights on $X$ with $U \leq V$. If $V > 0$ on $X$ and $CU_\infty(X, E)$ is closed in $FV_\infty(X, E)$, the Nachbin space $CV_\infty(X, E)$ is complete.
In case \( E \) is \( R \) or \( C \), the above theorem was obtained by Summers, under the hypothesis that \( U>0 \) on \( X \). (See Theorem 3.6 of [10].)

**Theorem 2.** Suppose that \( E \) is complete and \( U \) and \( V \) are two directed families of weights on \( X \) with \( U \leq V \). If \( V>0 \) on \( X \) and \( CU_{\omega}(X, E) \) is quasi-complete, the Nachbin space \( CV_{\omega}(X, E) \) is quasi-complete.

3. **Dual spaces** [6]. Throughout this paragraph \( X \) will be a locally compact Hausdorff space. In this case, for any set of weights \( V \) on \( X \), the space \( \mathcal{K}(X, E) \) is densely contained in \( CV_{\omega}(X, E) \). In fact, even \( \mathcal{K}(X) \otimes E \) is densely contained in \( CV_{\omega}(X, E) \). Let \( E'_{\omega} \) denote the topological dual of \( E \) endowed with the topology \( \sigma(E', E) \). An \( E'_{\omega} \)-valued bounded Radon measure \( u \) on \( X \) is a continuous linear mapping \( u \) from \( \mathcal{K}(X) \) into \( E'_{\omega} \) when \( \mathcal{K}(X) \) is endowed with the topology of uniform convergence on \( X \). Following Grothendieck [4], an \( E'_{\omega} \)-valued bounded Radon measure \( u \) on \( X \) is called integral if the linear form \( L \) defined over \( \mathcal{K}(X) \otimes E \) by \( L(\sum \phi_i \otimes y_i) = \sum \langle y_i, u(\phi_i) \rangle \) is continuous in the topology induced by \( C_{\omega}(X, E) \), in which case it can be uniquely continuously extended to \( C_{\omega}(X, E) \)'; if we define \( u(\phi) \) for each \( \phi \in \mathcal{K}(X) \) by \( \langle y, u(\phi) \rangle = L(y \otimes \phi) \) for all \( y \in E \), then \( u \) is an \( E'_{\omega} \)-valued bounded Radon measure. The transpose \( u' \) of \( u \) is a linear map from \( E \) into \( M_b(X) \), the space of all bounded Radon measures on \( X \). For every \( y \in E \) there corresponds a unique regular Borel measure \( \mu_y \) such that \( \mu_y(B) = \langle u'(y), \chi_B \rangle \), for all Borel subsets \( B \) of \( X \). There exists a continuous seminorm \( p \) on \( E \) and a constant \( k>0 \) such that \( |L(f)| \leq k\|f\|_p \) for all \( f \in C_{\omega}(X, E) \). Hence \( |\langle y, u(\phi) \rangle| = |L(y \otimes \phi)| \leq kp(y) \|\phi\|_{\omega} \). Thus, the bounded Radon measure \( u'(y) \) has norm \( \|u'(y)\| \leq kp(y) \), and the corresponding Borel measure \( \mu_y \) is such that \( |\mu_y(B)| \leq \|\mu_y\| \leq kp(y) \). This shows that, for a fixed Borel subset \( B \subset X \), the map \( y \mapsto \mu_y(B) \) belongs to \( E' \). Call this map \( \mu(B) \). The set function \( B \mapsto \mu(B) \), defined on the \( \sigma \)-ring of all Borel subsets of \( X \) and with values on \( E' \), is countably additive. For any finite families \( \{B_i\}_{i \in I} \) of disjoint Borel subsets of \( X \), whose union is \( X \), and \( \{y_i\}_{i \in I} \) of elements of \( E \) with \( p(y_i) \leq 1 \) for each \( i \in I \), we have

\[
(*) \quad \left| \sum_{i \in I} \langle y_i, \mu(B_i) \rangle \right| \leq k.
\]

An \( E'_{\omega} \)-valued bounded Radon measure \( u \) on \( X \) such that the corresponding set function \( \mu \) satisfies (*) for some continuous seminorm \( p \) on \( E \) and some constant \( k>0 \) is said to have finite \( p \)-semivariation. On the other hand, following Dieudonné [2], an \( E'_{\omega} \)-valued bounded bounded
Radon measure on $X$ is said to be \textit{$p$-dominated} if there is a positive bounded Radon measure $\mu$ on $X$ such that $|\langle y, u(\phi) \rangle| \leq \mu(\phi)p(y)$ for all $y \in E$ and $\phi \in \mathcal{K}(X)$. The arguments contained in Singer [9] and Câc [1] can be extended to prove the following:

**Lemma.** Let $u$ be an $E'_w$-valued bounded Radon measure on $X$. The following are equivalent:

(a) $u$ is integral;

(b) $u$ is $p$-dominated, for some continuous seminorm $p$ on $E$;

(c) $u$ has finite $p$-semivariation, for some continuous seminorm $p$ on $E$.

We denote by $M_b(X, E')$ the vector space of all $E'_w$-valued bounded Radon measures on $X$ which satisfy (a) or (b) or (c).

**Theorem 3.** Let $CV_\omega(X, E)$ be a Nachbin space. Then $VM_b(X, E')$ is linearly isomorphic to $CV_\omega(X, E')$.

4. Bishop's generalized Stone-Weierstrass theorem [7]. If $A$ is a subalgebra of $C(X)$, a subset $K \subset X$ is said to be \textit{antisymmetric} for $A$ if, for $f \in A$, the restriction $f|_K$ being real-valued implies that $f|_K$ is constant. Every antisymmetric set for $A$ is contained in a maximal one, and the collection $\mathfrak{K}_A$ of maximal antisymmetric sets for $A$ forms a closed, pairwise disjoint covering of $X$ (Glicksberg [3]). The following form of Bishop's generalized Stone-Weierstrass theorem is valid for Nachbin spaces ($X$ is as in §3).

**Theorem 4.** Let $V \subset C^+(X)$ and let $A$ be a subalgebra of $C(X)$ such that every $g \in A$ is bounded on the support of every $v \in V$. Let $W$ be a vector subspace of $CV_\omega(X, E)$ which is an $A$-module. Then $f \in CV_\omega(X, E)$ is in the closure of $W$ if and only if $f|_K$ is in the closure of $W|_K$ in $CV_\omega(K, E)$ for each $K \in \mathfrak{K}_A$.

If $E$ is $\mathbf{R}$ or $\mathbf{C}$ the hypothesis $V \subset C^+(X)$ can be strengthened to $V \subseteq C^+(X)$. If $A$ is selfadjoint, the conclusion of Theorem 4 is that $W$ is localizable under $A$ in $CV_\omega(X)$ (see Definition 4, Nachbin [5]). Let $CV_\omega(X, E)$ be an $A$-module, where $A$ satisfies the hypothesis of Theorem 4 and its maximal antisymmetric sets are sets reduced to a point, (e.g., $C_b(X)$, the algebra of all bounded continuous complex-valued functions). Under this hypothesis the following spectral synthesis result holds.

**Theorem 5.** Every proper closed $A$-submodule $W \subset CV_\omega(X, E)$ is contained in some closed $A$-submodule of codimension one in $CV_\omega(X, E)$ and is the intersection of all proper closed $A$-submodules of codimension one in $CV_\omega(X, E)$ which contain it.
5. **Dieudonné theorem for density in tensor products of Nachbin spaces** [8]. Let $X$ and $Y$ be two completely regular Hausdorff spaces and $V$ and $W$ two directed families of weights on $X$ and $Y$ respectively. Let $V \times W$ denote the set of all functions $(x, y) \mapsto v(x)w(y)$ on $X \times Y$. Let $A$ be a locally convex topological algebra and let $E$ and $F$ be two locally convex spaces which are topological modules over $A$. Then $E \otimes_A F$ is defined to be the quotient space $(E \otimes F)/D$, where $E \otimes F$ has the projective tensor product topology and $D$ is the closed linear span of the elements of the form $au \otimes v - u \otimes av$, where $a \in A$, $u \in E$, $v \in F$. If $f \in CV_\omega(X, E)$ and $g \in CW_\omega(Y, F)$, then $f \otimes_A g$ belongs to $C(V \times W)_\omega(X \times Y, E \otimes_A F)$, where $f \otimes_A g$ denotes the map $(x, y) \mapsto f(x) \otimes_A g(y)$.

**Theorem 6.** The vector subspace of all finite sums of mappings of the form $f \otimes_A g$, where $f \in CV_\omega(X, E)$ and $g \in CW_\omega(Y, F)$, is dense in $C(V \times W)_\omega(X \times Y, E \otimes_A F)$.

**References**


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