PL CHARACTERISTIC CLASSES AND COBORDISM

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1. Introduction. In this note, we announce results on the structure of the unoriented PL cobordism ring, $\mathcal{M}_*$, and the $\mathbb{Z}_2$-characteristic classes for PL bundles, $H^*(BPL)$. All homology and cohomology is with $\mathbb{Z}_2$-coefficients, unless otherwise indicated.

There is a sequence of $H$-space fibrations

$$\Omega(G/PL) \to PL \to G \to G/PL \to BPL \to BG.$$ 

The $\mathbb{Z}_2$-cohomology of an $H$-space is a Hopf algebra over the Steenrod algebra. R. J. Milgram [5] has determined $H^*(G)$ and $H^*(BG)$, and D. Sullivan [7] has determined $H^*(G/PL)$ and $H^*(\Omega(G/PL))$. Our main results, determining the Hopf algebra structure of $H^*(PL)$ and $H^*(BPL)$, follow from spectral sequence arguments, once we have determined the map $H^*(G/PL) \to H^*(G)$.

W. Browder, A. Liulevicius, and F. P. Peterson [1] have shown that there is an isomorphism of rings $\mathcal{M}_*^{PL} \cong \mathcal{M}_*^{TOP} \otimes H^*(BPL)//H^*(BO)$, where $\mathcal{M}_*^{PL}$ is the unoriented, differentiable cobordism ring determined by Thom. Thus our homology computations are sufficient to determine $\mathcal{M}_*^{PL}$.

Our methods also determine $H^*(TOP)$ and $H^*(BTOP)$ as Hopf algebras. In fact, these computations are easier than the PL computations. The Kirby-Siebenmann topological transversality theorem implies that $\mathcal{M}_*^{TOP} \cong \pi_*(MTOP) = \mathcal{M}_*^{TOP} \otimes H^*(BTOP)//H^*(BO)$ in dimensions $\neq 4$.


The homotopy groups are given by $\pi_n(G/PL) = \pi_n(G/TOP) = P_n$, where $P_n = \mathbb{Z}$, 0, $\mathbb{Z}_2$, 0 as $n \equiv 0, 1, 2, 3 \pmod{4}$, respectively. However, the natural map $G/PL \to G/TOP$ has as fibre an Eilenberg-Mac Lane space $K(\mathbb{Z}_2, 3)$.

There is a map $G/TOP \to \prod_{n \geq 1} K(P_n, n) = K(P_*)$ which induces an isomorphism of Hopf algebras over the Steenrod algebra $H^*(K(P_*)) \cong H^*(G/TOP)$. Let $k_2 \in H^{2n}(G/TOP)$ denote the image of the fundamen-
mental class $i_{2n} \in H^*(K(P_4))$. Then the map $i^*: H^*(G/TOP) \to H^*(G)$ is completely determined, once the elements $i^*(k_{2n}) \in H^{2n}(G)$ have been computed, $n \geq 1$.

**Remark 2.1.** Denote also by $k_{2n}$ the image of $k_{2n}$ in $H^{2n}(G/PL)$. Since $G/PL$ has one nonzero $k$-invariant, $\delta k_2 \in H^2(K(Z_2, 2), Z)$, where $\delta$ is the integral Bockstein, it follows that $k_4 = k_2^2 \in H^4(G/PL)$. However, since $\delta k_2$ is divisible by 2, $H^*(G/PL)$ and $H^*(G/TOP)$ are abstractly isomorphic as algebras over the Steenrod algebra. Thus $H^*(G/PL)$ has generators $\{k_{2n}, n \neq 2, k_4\}$ where $k_4 \in H^4(G/PL)$ is a new generator. The Hopf algebra structure of $H^*(G/PL)$ is determined by the coproduct $\Delta(k_4) = k_4 \otimes 1 + k_2 \otimes k_2 + 1 \otimes k_4$.

Let $M^m$ be a closed manifold, $m \equiv 0 \pmod 2$, and let $\phi: M^m \to G/PL$ be a map. Then there is a Kervaire surgery obstruction $s_K(M^m, \phi) \in \mathbb{Z}_2$ and a formula for $s_K$ which uniquely characterizes the class $K_{4n-2} = \sum n \geq 1 \ k_{4n-2}$ [6], [7] 2.2

\[ s_K(M^m, \phi) = \langle V^2(M) \cdot \phi^*(K_{4n-2}), [M] \rangle \in \mathbb{Z}_2 \]

where $V^2(M)$ is the square of the total Wu class $V(M) = \sum_{i \geq 0} V_i(M) \in H^*(M)$.

Let $M^m$ be a $\mathbb{Z}_2$-manifold (that is, $w_1(M)$ is the reduction of an integral class $w_1(M) \in H^1(M, Z)$), $m \equiv 0 \pmod 4$, and let $\phi: M^m \to G/PL$ be a map. Then there is an index surgery obstruction $s_I(M^m, \phi) \in \mathbb{Z}_2$ and a formula for $s_I$ which uniquely characterizes the class $K_{4*} = \sum n \geq 1 k_{4n}$.

\[ s_I(M^m, \phi) = \langle V^2(M) \cdot \phi^*(K_{4*}) \]

\[ + Sq^1((\sum V_{2i}(M)Sq^1V_{2i}(M))\phi^*(K_{4* - 2})), [M] \rangle \in \mathbb{Z}_2 \]

3. The homology of $SG$. A sequence $I = (i_1, \ldots, i_n)$ of positive integers is allowable if $2i_{j+1} \geq i_j$, all $j$. We write $kI = (ki_1, \ldots, ki_n)$, $d(I) = \sum_{j=1}^n i_j$, and $e(I) = i_1 - (\sum_{j=2}^n i_j)$. Let $S(n)$ denote the set of allowable sequences $I$ of length $n$ with $e(I) \geq 0$.

If $A$ is a graded Hopf algebra over $\mathbb{Z}_2$, let $A^*$ denote the dual Hopf algebra, and let $\Lambda(A) \subset A$ denote the Hopf subalgebra generated by squares.

If $X$ is a graded set, introduce Hopf algebras $P(X)$, the polynomial algebra on primitive generators $X$, $\Gamma(X) = P(X)^*$, the divided power algebra on $X$, and $E(X)$, the exterior algebra on primitive generators $X$. Then $E(X) \simeq E(X)^*$. The graded set $s(X)$ will be the set $X$ with elements shifted up one dimension.

The space $SG$ is studied in [4] and [5] by identifying it with the degree one component of $QS^0 = \lim_{n \to \infty} (\Omega^n S^n)$. If $x, y \in H_\bullet(SG)$, de-
note by \( x \cdot y \in H_*(SG) \) their composition product, and denote by \( x \ast y \in H_*(SG) \) the loop product \( x \ast y \ast [-1] \), computed in \( H_*(QS^0) \). Here \([q]\) denotes the homology class of a point in the degree \( q \) component of \( QS^0 \). Let \( Q^f = Q^i \circ \cdots \circ Q^s \) be the Dyer-Lashof operation. If \( I \in S(n) \), let \( e_I = Q^f[1] * [1 - 2^n] \in H_{4dI}(SG) \). The notation is that of [4]. The following two paragraphs and Lemma 3.1 are reformulations of theorems of [5].

There is an isomorphism of Hopf algebras
\[
H_*(SG) \cong H_*(SO) \otimes A \otimes (\otimes_{n \geq 1} C_n) \text{ where } A = \mathbb{Z}_2[e_{i, i} | i \geq 1] \text{ and } C_n = \mathbb{Z}_2[e_I | I \in S(n), e(I) \geq 1] \text{ are Hopf subalgebras of } H_*(SG).
\]

As an algebra, \( H_*(SO) \cong E(e_{i, i} | i \geq 1) \). The coproduct is \( \Delta(e_n) = \sum_{i+j=n} e_i \otimes e_j \). Further, \( e_i = e_*([RP(i)]) \), where \( :RP(\infty) \to SO \) is a certain map.

There is an isomorphism of Hopf algebras \( H_*(BG) = H_*(BO) \otimes BA \otimes BC_2 \otimes (\otimes_{n \geq 1} \overline{BC_n}) \), where \( BA = E(s(e_{i, i}) | i \geq 1) \), \( BC_n = P(s(e_I | I \in S(n), e(I) \geq 1)) \), and \( \overline{BC_n} = P(s(e_I | I \in S(n), e(I) \geq 2)) \).

**Lemma 3.1.** If \( x \in H_*(SG) \), then \( x = \lambda(x)e_n + \sum y'_i \cdot y'_i + \sum z'_i \ast z'_i \), where \( \lambda(x) = 0 \) or 1 and \( y'_i, y'_i, z'_i, z'_i \in H_*(SG) \) are elements of positive dimensions. In particular, the classes \( e_i \) generate \( H_*(SG) \) if both products \( \ast \) and \( \ast \) are used.

Next, we need a geometric interpretation of the loop product in \( H_*(SG) \). Let \( x, y \in H_*(SG) \) be represented by manifolds \( \alpha : M^a \to SG \) and \( \beta : N^b \to SG \). Then \( \alpha \ast [-1] : M^a \to QS^0 \) corresponds to a map \( M^a \times S^a \to S^a \) of degree zero on \( p \times S^4 \), \( p \in M \). By transverse regularity, this, in turn, corresponds to a degree zero map \( f : M' \to M \) covered by a bundle map \( \tilde{f} : \nu_M \to \nu_M \). Similarly, let \( \beta : N^b \to SG \) correspond to a degree zero map \( g : N' \to N \), covered by a bundle map \( \tilde{g} : \nu_N \to \nu_N \).

**Lemma 3.2.** The element \( x \ast y \in H_*(SG) \) is represented by a map 
\( \alpha \ast \beta : M \times N \to SG \), which corresponds to the degree one normal map
\[
M \times N + M' \times N + M \times N' \xrightarrow{1 + (f \times 1) + (1 \times g)} M \times N,
\]
covered by the bundle map
\( \hat{1} + (\hat{f} \times \hat{1}) + (\hat{1} \times \hat{g}) \),

where \( + \) indicates disjoint union of manifolds.

4. The map \( H^*(G/PL) \to H^*(SG) \).

**Theorem 4.1.** Let \( \alpha : M^a \to SG \) and \( \beta : N^b \to SG \) be maps, \( a + b = 2n \). Then
\[ s_K(M \times N, \alpha - \beta) = s_K(M \times N, \alpha \cdot \beta) \]
\[ = \langle (V(M \times N) \cdot \alpha^* \sigma(V) \otimes 1)_n, (V(M \times N) \cdot 1 \otimes \beta^* \sigma(V))_n, [M \times N] \rangle \]
\[ = \left\langle V^2(M \times N), \sum_{r \neq 2} \sum_{i+j=2, i \neq j \neq 2} \alpha^* \sigma(w_i) \otimes \beta^* \sigma(w_j) \right\rangle, [M \times N] \]
\[ \in \mathbb{Z}_2 \]

where \( \sigma(w_i) \in H^{i-1}(SG) \) is the suspension of \( w_i \in H^i(BS_G) \).

**Theorem 4.2.** Let \( \alpha: M^a \to SG \) and \( \beta: N^b \to SG \) be maps, \( a+b=4n \), where \( M^a \) and \( N^b \) are \( \mathbb{Z}_2 \)-manifolds. Then
\[
\begin{align*}
  &s_i(M \times N, \alpha - \beta) = s_i(M \times N, \alpha \cdot \beta) \\
  = &\langle Sq^1((V(M \times N) \cdot \alpha^* \sigma(V) \otimes 1)_{2n-1}), [M \times N] \rangle \\
  = &\left\langle V^2(M \times N) \left( \sum_{i \geq 1} \sigma(w_i)^{2i} \otimes \sigma(w_i)^{2i} \right) \\
  &+ Sq^1 \left( \left( \sum_{i \geq 0} V_{2i}(M) Sq^1 V_{2i}(M) \right) \left( \sum_{r \neq 2} \sum_{i+j=2, i \neq j \neq 2} \alpha^* \sigma(w_i) \otimes \beta^* \sigma(w_j) \right) \right), [M \times N] \right\rangle \\
  \in &\mathbb{Z}_2.
\end{align*}
\]

Theorem 4.1 is proved using Lemma 3.2, and the result of E. H. Brown, Jr., that the Kervaire surgery obstruction of a degree one normal map may be expressed as a difference of two Arf invariants [2]. To compute this difference in the situation of Lemma 3.2, an additional formula of Brown is needed, which expresses how the Arf invariant of a manifold \( M^{2n} \) depends on the choice of a degree one map \( S^{2n} \to T(p_M^*) \). The second equality in Theorem 4.1 is a lengthy computation with Stiefel-Whitney numbers. It is first verified for the products \( e_a \cdot e_b: RP(a) \times RP(b) \to SG \), and then the general case is deduced as a corollary.

The proof of Theorem 4.2 is similar to the proof of 4.1, once analogues of Brown’s results for the index surgery obstruction for \( \mathbb{Z}_2 \)-manifolds have been established.

As consequences of 2.2, 3.1, and 4.1, and 2.3, 3.1, and 4.2, we have

**Theorem 4.3.** Let \( k_{4n-2} \in H^{4n-2}(G/TOP) \) be as in §2. Then \( i^*(k_{4n-2}) = 0 \in H^*(SG) \) if and only if \( 4n \neq 2^i \). If \( 4n = 2^i \), then \( \langle i^*(k_{2^i-2}), e_T \rangle = 1 \) if and only if \( I \in S(2), d(I) = 2^i - 2 \).
Theorem 4.3 was first proved by Madsen, using the techniques of \cite{3}.

**Theorem 4.4.** Let \( k_{4n} \in H^{4n}(G/TOP) \) be as in \S 2. Then \( i^*(k_{4n}) = 0 \in H^*(SG) \) if \( 4n \neq 2^j \). If \( 4n = 2^j \), then \( i^*(k_{2^j}) = i^*(k_{2^{j-1}}) \). Hence \( i^*(k_{2^j} + k_{2^{j-1}}) = 0 \).

**Remark 4.5.** Note that by Remark 2.1 and Theorems 4.3 and 4.4, the map \( i^*: H^*(G/PL) \to H^*(SG) \) is also computed since \( \langle i^*(k_{2^j}), \bar{e}_{(1,1)} \rangle = 1 \).

Let \( K(P_*) = K_1 \times K_2 \) where \( K_1 = \prod_{n=2^j+2} K(P_n, n) \) and \( K_2 = \prod_{n=2^j+2} K(P_n, n) \).

**Theorem 4.6.** There is an exact sequence of Hopf algebras

\[
\mathbb{Z}_2 \to H_*(SO) \otimes \Lambda(A \otimes \mathbb{C}_2) \otimes \left( \bigotimes_{n \geq 3} C_n \right) \to H_*(SG) \\
\to H_*(G/TOP) \to \Gamma(W) \otimes H_*(K_2) \to \mathbb{Z}_2
\]

where \( W \) is a graded set such that there is an isomorphism of Hopf algebras \( H_*(K_1) \cong \Gamma(W) \otimes \Gamma(I \mid I \subseteq S(2), I \neq 2J) \).

5. The main theorems. In this section, we state the main results. The proofs consist of (careful) applications of the Eilenberg-Moore or Serre spectral sequence of the fibrations involved.

**Theorem A.** There is an isomorphism of Hopf algebras

\[
H_*(BTOP) \cong H_*(BO) \otimes BC_3 \otimes \left( \bigotimes_{n \geq 4} BC_n \right) \otimes E(s(2I \mid I \subseteq S(2))) \\
\otimes \Gamma(W) \otimes H_*(K_2).
\]

Further, \( H_*(BO) \otimes BC_3 \otimes \left( \bigotimes_{n \geq 4} BC_n \right) \cong \text{image } (H_*(BTOP) \to H_*(BG)) \), and \( \Gamma(W) \otimes H_*(K_2) \cong \text{image } (H_*(G/TOP) \to H_*(BTOP)) \).

**Theorem B.** There is an isomorphism of Hopf algebras

\[
H_*(STOP) \cong H_*(SO) \otimes \Lambda(A \otimes C_2) \otimes \left( \bigotimes_{n \geq 3} C_n \right) \otimes \Gamma(V) \otimes H_*(\Omega K_2)
\]

where \( V \) is a graded set such that, as algebras, \( \Gamma(V) = E(s^{-1}(W)) \).

The computation of \( H_*(BPL) \) is more complicated because of Remarks 2.1 and 4.5. First, we need more notation. Let

\[
V = \{ 2^i(2, 1, 1) \mid i \geq 0 \} \cup \{ 2^j(2^{i+1} + 1, 2^j + 1, 2^j) \mid i, j \geq 0 \} \\
\cup \{ 2^i(2^{j+k+1} + 2^j + 1, 2^{j+k} + 2^j, 2^{j+k}) \mid i, j, k \geq 0 \} \subseteq S(3).
\]
Let \( X = S(3) - Y \), and let \( X_0 = \{ I \in X \mid e(I) = 0 \} \). Let \( X_1 = X - X_0 \). Finally, let \( K'_2 = \prod_{n=1,2^l=2} K(P_n, n) \).

**Theorem C.** There is an isomorphism of Hopf algebras

\[
H_\ast(BPL) \cong H_\ast(BO) \otimes P(Z) \otimes \left( \bigotimes_{n \geq 4} BC_n \right) \otimes P(s(X_1)) \otimes \Gamma(W) \otimes H_\ast(K'_2) \otimes E(s(X_0)) \otimes E(s(2I \mid I \in Y)).
\]

where \( Z \) is a graded set such that \( P(Z) \otimes \Lambda(P(s(X_1))) \cong BC_4 \).

**Theorem D.** There is an isomorphism of Hopf algebras

\[
H_\ast(SPL) \cong H_\ast(SO) \otimes \left( \bigotimes_{n \geq 4} C_n \right) \otimes Z_2[ci \mid I \in X] \otimes \Lambda(Z_2[ci \mid I \in Y]) \otimes \Gamma(V) \otimes H_\ast(\Omega K'_2).
\]

**Remark.** It is easy to read off the dual Hopf algebras \( H^\ast(BTOP) \) and \( H^\ast(BPL) \) and the cobordism ring \( \Theta^\ast \cong \Theta^\ast \otimes (H^\ast(BPL) / H^\ast(BO)) \) from Theorems A and C.

**References**


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