EXTENSION OF POSITIVE HOLOMORPHIC LINE BUNDLES

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Communicated by Gian-Carlo Rota, May 7, 1971

In this note, we announce a result on extending complex line bundles through subvarieties of codimension 2. The motivation for this result is that it allows us to extend a recent result of Phillip Griffiths [2] on meromorphically extending holomorphic maps into compact Kahler manifolds. Details and further related results will appear elsewhere.

A holomorphic line bundle $L$ on a complex manifold $M$ is said to be semipositive if there exists a hermitian metric $h$ on $L$ such that the curvature form

$$\Omega = \frac{i}{2\pi} \partial \bar{\partial} \log h$$

is positive semidefinite at all points of $M$ (i.e., the locally defined functions $\log h$ are plurisubharmonic).

**Theorem.** Let $M$ be a complex manifold, and let $S$ be an analytic set in $M$ such that $\text{codim } S = 2$. Then every semipositive holomorphic line bundle $L$ on $M - S$ extends to a holomorphic line bundle on $M$.

If $\text{codim } S \geq 3$, then it is a well-known fact that any line bundle $L$ on $M - S$ extends to $M$ (see [3]).

In order to prove the theorem, one must show that $L$ induces the zero element of $H^2(S, O^*) = \Gamma(M, \mathcal{O}_M^\otimes \mathcal{O}^*)$. Therefore it suffices to show that $L$ extends locally, and the theorem is then a consequence of the following lemma applied to the curvature form $\Omega$.

We let $D$ denote the open unit disk in $\mathbb{C}$.

**Lemma.** Let

$$\omega = i \sum f_{\alpha\beta} dz_{\alpha} \wedge d\bar{z}_{\beta}, \quad (1 \leq \alpha, \beta \leq n)$$


Key words and phrases. Holomorphic line bundle, curvature form, positive line bundle, sheaf cohomology, local cohomology, harmonic function, plurisubharmonic function, analytic subvariety, Kahler manifold, meromorphic map.

1 This research was partially supported by National Science Foundation Grant GP-21193.
be a real closed \((1, 1)\)-form on the domain

\[ W = (D^2 - 0) \times D^{n-2} \subset \mathbb{C}^n. \]

If \( f_{11} \geq 0 \) and \( f_{22} \geq 0 \) on \( W \), then there exists a real-valued function \( u \) on \( W \) such that \( \omega = dd^c u \). In particular, if \( \omega \) is a Kahler form on \( W \), then \( \omega = dd^c u \), where \( u \) is a function on \( W \).

Write \( W = W_1 \cup W_2 \), where

\[ W_j = \{ z \in W : z_j \neq 0 \}, \quad \text{for } j = 1, 2. \]

Then \( \omega \mid W_j = dd^c u_j \), where \( u_j \) is a real-valued function on \( W_j (j = 1, 2) \). Let \( h = u_1 - u_2 \) on \( W_1 \cap W_2 \). Then \( dd^c h = 0 \), i.e., \( h \) is pluriharmonic. For the case \( n = 2 \), \( u_1 \) and \( u_2 \) are subharmonic in each variable separately. The main point of the lemma (and the theorem) is that we can then write \( h = h_1 - h_2 \), where \( h_j \) is a pluriharmonic function on \( W_j (j = 1, 2) \), and therefore \( u = u_j - h_j \) is a globally defined function on \( W \) with \( \omega = dd^c u \). The proof uses the solution of the Dirichlet problem on the annulus \( A_r = \{ r < |z| < 1 \} \). By considering the biannulus \( A_r \times A_r \) and letting \( (r, s) \to (0, 0) \), one constructs functions \( \bar{u}_j \) on \( W_j (j = 1, 2) \) such that \( \bar{u}_1 \) and \( \bar{u}_2 \) are harmonic in each variable separately, and \( h = \bar{u}_1 - \bar{u}_2 \). The existence of \( h_1 \) and \( h_2 \) then follows from the equation \( h = \bar{u}_1 - \bar{u}_2 \).

In [2], Phillip Griffiths proved the following result.

**Theorem (Griffiths).** Let \( f : D^n - 0 \to X \) be a holomorphic map, where \( X \) is a compact Kahler manifold. If \( n \geq 3 \), then \( f \) extends meromorphically to \( D^n \) (i.e., the closure of the graph of \( f \) is an analytic set in \( D^n \times X \)).

Griffiths' idea is to apply a theorem of Errett Bishop [1], [4] on extending analytic sets with finite volume. Consider the Kahler form

\[ \omega = \frac{i}{2} \sum d\bar{z}_a \wedge dz_a + f^* \omega_X \]

on \( D^n - 0 \), where \( \omega_X \) is the given Kahler form on \( X \). The volume of the graph of \( f \) is then given by \( \int \omega^n \). Since \( H^2(D^n - 0, \mathbb{R}) = 0 \) and \( H^1(D^n - 0, \mathbb{C}) = 0 \), for \( n \geq 3 \), one can write \( \omega = dd^c u \), where \( u \) is a plurisubharmonic function on \( D^n - 0 \). By approximating \( u \) by smooth plurisubharmonic functions on a ball \( B \subset D^n \) about 0 and by applying Stokes' theorem, Griffiths concludes that \( \int_B (dd^c u)^n < +\infty \).

By the above lemma, we can also write \( \omega = dd^c u \) for the case \( n = 2 \) (although \( H^1(D^2 - 0, \mathbb{C}) \neq 0 \)). Therefore, Griffiths' theorem is also valid for \( n = 2 \).
REFERENCES


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