ANALYTIC FUNCTIONS WITH UNIVALENT DERIVATIVES
AND ENTIRE FUNCTIONS OF EXPONENTIAL TYPE

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Abstract. Functions $f$, analytic and univalent in the unit disc, and such that all successive
derivatives $f^{(k)}$ are univalent in this disc, are necessarily transcendental entire functions
of exponential type. These functions, and functions $f$ having an infinite number of derivatives
$f^{(k)}$ univalent in the unit disc, are discussed. Entire functions of bounded index are of
exponential type and their properties are also discussed.

1. Introduction. Let $f(z)$ be analytic in the unit disc $D: |z| < 1$. We say
that $f$ is univalent in $D$ if for each pair of distinct points $z_1, z_2$ in $D,
f(z_1) \neq f(z_2)$. In §§1–4 we give a brief survey of functions analytic and univalent in $D$. Functions $f$ such that $f(z)$ and each successive derivative
$f^{(k)}(z)$ are univalent in $D$ are considered next in §5. Such functions $f$ must
be transcendental entire functions of exponential type. Related problems of functions $f$ such that $f(z)$ and a sequence of derivatives $f^{(m)}(z)$ are
univalent or of functions $f$ such that $f(z)$ is entire and $f^{(k)}(z)$ is univalent in $|z| < \rho_k$ ($\rho_k > 0$) are considered in §§6–10. This is followed by a section
§11) on multivalent functions and three sections (§§12–14) on functions
of bounded index. An entire function $f(z)$ is said to be of bounded index
if there exists an integer $N$, independent of $z$, such that

$$
\max_{0 \leq s \leq N} \left\{ \frac{|f^{(s)}(z)|}{s!} \right\} \geq \frac{|f^{(j)}(z)|}{j!},
$$

for $j = 1, 2, \ldots$ and for all $z$. The smallest such integer $N$ is called the
index of $f$. An entire function $f$ of bounded index $N$ is of exponential type
not exceeding $(N + 1)$. Finally we mention some unsolved problems.

2. Conditions for the univalence of $f$. Let

$$
f(z) = \sum_{0}^{\infty} a_n z^n, \quad |z| < 1.
$$

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1 In this article we shall not consider meromorphic univalent functions.
ANALYTIC FUNCTIONS AND ENTIRE FUNCTIONS

If $a_1 \neq 0$ and

$$\sum_{n=2}^{\infty} n|a_n| \leq |a_1|, \tag{2.2}$$

then $f$ is analytic and univalent in $D$ and continuous on the closure of $D$. To prove this, let $z_1, z_2 \in D$, $\max_{i=1,2}|z_i| = r < 1$, $z_1 \neq z_2$. Then

$$\frac{|f(z_2) - f(z_1)|}{|z_2 - z_1|} = |a_1 + \sum_{n=2}^{\infty} a_n(z_2^{n-1} + z_2^{n-1}z_1 + \cdots + z_1^{n-1})| \geq |a_1| - \sum_{n=2}^{\infty} n|a_n|r^{n-1} > 0.$$

This implies that $f$ is univalent in $D$. Further, for every $N \geq 1$,

$$\sum_{n=2}^{N} n|a_n| \leq |a_0| + |a_1| + \sum_{n=2}^{\infty} \frac{n|a_n|}{n} \leq |a_0| + \frac{3|a_1|}{2},$$

and continuity of $f$ follows.

If the radius of convergence of the series in (2.1) defining $f$ is $R$, then $f$ is univalent in $|z| < \rho \leq R$, if $a_1 \neq 0$ and

$$\sum_{n=2}^{\infty} n|a_n|\rho^{n-1} \leq |a_1|. \tag{2.3}$$

Let $f$ be analytic in $D$. If $f$ is univalent in $D$ then $f'(z) \neq 0$ in $D$ [32, p. 23]. If

$$\Re(af'(z)) > 0, \quad z \in D, \tag{2.4}$$

for some complex number $a$, $|a| = 1$, then $f$ is univalent in $D$. This follows immediately from the following integral expression

$$\Re \left\{ \frac{a(f(z_2) - f(z_1))}{z_2 - z_1} \right\} = \int_{0}^{1} \Re \{af'((1 - w)z_1 + wz_2)\} \, dw.$$

Another criterion for univalence of $f$ ([62]; see also [29]) is as follows. Let

$$\{w, z\} = \left( \frac{f''}{f'} - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 \right)$$

be the Schwarzian derivative of $w = f(z)$ with respect to $z$. In order that $w = f(z)$ be univalent in $D$ it is necessary that

$$|(w, z)| \leq 6/(1 - |z|^2)^2$$

and sufficient that

$$|(w, z)| \leq 2/(1 - |z|^2)^2.$$
Becker [2] has recently proved that \( f \) is univalent in \( D \) if
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{1}{(1 - |z|^2)}.
\]

3. **Class S.** Let \( S \) denote the collection of functions \( f \) analytic and univalent in \( D \) and normalized by the conditions \( f(0) = 0, f'(0) = 1 \). Thus \( f \in S \) can be written as
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad |z| < 1.
\]

Bieberbach [6] proved in 1916 that, for \( f \in S \),
\[
|a_2| \leq 2
\]
with equality if and only if
\[
f(z) = K_\alpha(z) = z/(1 - z e^{i\alpha})^2 \quad (\alpha \text{ real}).
\]

This function \( K_\alpha \) (Koebe function) maps \( D \) on the whole plane slit radially from \( w = -\frac{1}{2} e^{-i\pi/2} \) to infinity. It is extremal not only for \( a_2 \) but also for a number of other problems. Since \( |a_n| = n, n = 2, 3, \ldots \) for this function \( K_\alpha \), it was conjectured that, for \( f \in S \),
\[
|a_n| \leq n, \quad n = 2, 3, \ldots,
\]
with equality only for the Koebe function. This conjecture, called the Bieberbach conjecture, was proved for \( n = 3 \) by Loewner [58] in 1923, for \( n = 4 \) by Charzynski and Schiffer ([18]; see also [30]) in 1960, and for \( n = 6 \) by Pederson [66] in 1968 and Ozawa [64] in 1969 independently of each other. Garabedian and Schiffer [31] proved that (3.4) holds for a function \( f \in S \) which is “close enough” to the Koebe function and Aharonov has shown (3.4) to hold if \( |a_2| < 0.867 \) ([1]; see also [9]).

For each fixed \( f \in S \), Hayman (see [38, pp. 112–113]) has shown that \( |a_n| \leq n \) \((n > n_0(f))\). For all \( n \geq 2 \), Littlewood proved in 1925 (see [38, p. 10]) that \( |a_n| < en \). This estimate has recently been improved to \( |a_n| < 1.081n \) \((n \geq 2)\) by Carl H. Fitzgerald (see also [32, p. 612]).

4. **Subclasses of S.** A function \( f \in S \) is said to be starlike univalent in \( D \), or briefly starlike in \( D \) if \( f(D) \) is starlike with respect to the origin \( w = 0 \). A necessary and sufficient condition for \( f \in S \) to be starlike in \( D \) is that
\[
|a_n| \leq n, \quad n = 2, 3, \ldots,
\]
with equality only for the Koebe function. From (4.1) it is easy to obtain, for \( f \in S^* \), the following integral representation formula.
\[ f'(z) \equiv \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} dV(t) \]

where \( V(t) \) is an increasing function of \( t \), \( V(t) - t \) has period \( 2\pi \) and \( (1/2\pi) \int_0^{2\pi} dV(t) = 1 \). A second subclass of \( S \) is the class of convex univalent functions. We say that \( f \in S \) is convex univalent in \( D \) if \( f(D) \) is a convex set. We denote this subclass of \( S \) by \( K \). A necessary and sufficient condition for \( f \in S \) to be in \( K \) is that \([38, pp. 140-141], [32, p. 166]\),

\[ \text{Re}(1 + zf''(z)/f'(z)) > 0, \quad |z| < 1. \]

If \( f \in K \) then \( |a_n| \leq 1 \). If \( f \in S^* \) then \( |a_n| \leq n \).

A third subclass of functions is the class of close-to-convex functions introduced by Kaplan \[46\]. A function \( f \in S \) is close-to-convex if and only if

\[ \text{Re}(f'(z)/\phi'(z)) > 0, \quad |z| < 1, \]

where \( \phi(z)/\phi'(0) \in K \). (If \( f \) is analytic in \( D \) and satisfies the close-to-convex condition (4.4) then it is univalent.) For this class (3.4) also holds. If \( f \) is defined by (2.1) and satisfies (2.2) and if \( f(0) = 0, f'(0) \neq 0 \), then \( f \) is starlike in \( D \) \[33\]. From this we can conclude that if \[33\]

\[ \sum_{k=2}^{\infty} k^2|a_k| \leq |a_1| \]

then \( f \) is convex in \( D \).

For more information on various problems of univalent function theory we refer the reader to five excellent survey articles by Bernardi \[3\], Hayman \[39\], Goluzin \[32, pp. 577-628\], Goodman \[35\] and Robertson \[73\]. We list some recent papers in the bibliography at the end and refer to an exhaustive bibliography by Bernardi \[4\], for books and periodical literature up to 1965.

5. Functions with univalent derivatives. Let \( f \in S \) and let \( E \) denote the subclass

\[ E = \{ f | f \in S, f^{(k)} \text{ is univalent in } D \text{ for } k = 1, 2, \ldots \} \]

If \( f \in E \) then \( f \) must be a transcendental entire function of exponential type, that is,

\[ \limsup_{r \to \infty} \frac{\log M(r, f)}{r} = T^* < \infty, \]

where as usual \( M(r, f) = \max_{|z|=r} |f(z)| \). (Note that functions, for which \( 0 \leq T^* < \infty \), and in particular functions of order less than one, are all
functions of exponential type.) More precisely we have \[84\]

\begin{equation}
|f(z)| \leq \frac{\exp(2\alpha|z|) - 1}{2\alpha},
\end{equation}

where \(\alpha = \sup\{|a_2| : f \in E\}\) and

\begin{equation}
\pi/2 \leq \alpha < 1.7208.
\end{equation}

To prove this we note that if \(f \in E\) then \(a_n + 1 \neq 0\). Define \(F_n\) in \(D\) by

\[F_n(z) = \frac{f^{(n)}(z) - n!a_n}{(n+1)!a_{n+1}}.\]

Then \(F_n \in E\) and we have

\[|a_{n+2}| \leq \frac{2\alpha|a_{n+1}|}{n + 2}.
\]

An inductive argument gives \(|a_n| \leq (2\alpha)^{n-1}/n!\) (\(n \geq 2\)). This implies that \(f\) is entire and satisfies (5.3). Since \(|a_2^2 - a_3| \leq 1 - (M(1))^{-2}\) ([44], [88]), we have

\[
\alpha^2 \leq 3\left(1 - 4\alpha^2/(e^{2\alpha} - 1)^2\right).
\]

This implies the right-hand inequality in (5.4). To complete the proof of (5.4) we observe that \(\phi(z) = (\exp(\pi z) - 1)/\pi \in E\) and \(a_2\) for this function is \(\pi/2\).

We note here that the property of univalence is only one of the properties which forces \(f\) to be entire. Consider a property (A) which a function analytic in \(D\) is able to possess. We say that (A) is an admissible property provided the following hold: (i) if \(f\) has (A) then \(f'(0) \neq 0\). (ii) If \(f\) has (A) and if \(b\) and \(c\) are complex numbers with \(b \neq 0\), then the function \(F(z) = bf(z) + c\) also has (A). Let \(T\) be the family of functions \(f\), analytic in \(D\), of the form (3.1). Let \(T(A)\) be the subclass of \(T\) such that if \(f \in T(A)\) then \(f^{(n)}\) has property (A) for \(n = 0, 1, 2, \ldots\). Suppose that \(T(A)\) is not empty and let \(\alpha_A = \sup\{|a_2| : f \in T(A)\}\). If \(\alpha_A < \infty\) and \(f \in T(A)\) then \(f\) is a transcendental entire function of exponential type not greater than \(2\alpha_A\) [86]. For instance one can take property (A) to be property (K). We say that \(f\) has (K) if \(f\) is convex univalent in \(D\). Then (K) is an admissible property. Further \(\alpha_K = \sup\{|a_2| : f \in T(K)\}\) lies between \(\frac{1}{2}\) and 0.6838 [86].

\[6.\textbf{Not all derivatives univalent.}\] Let \(f\) be defined in \(D\) by (2.1) and let \(\{n_k\}_{k=1}^{\infty}\) be a sequence of strictly increasing positive integers. Suppose that each \(f^{(n_k)}\) is univalent in \(D\). Let \(R\) be the radius of convergence of the series in (2.1). If the sequence \(\{n_k\}\) does not increase very rapidly, we may have \(R > 1\). Thus, for instance [86],
From (6.1) it is easy to show that if \( n_{k+1} - n_k = o(\log k) \) then \( R = \infty \) and \( f \) is entire. If \( n_{k+1} - n_k = O(1) \) then \( f \) is of exponential type.

A more general result of this type is as follows. Let \( \phi(x) \) and \( \theta(x) \) be two slowly oscillating functions (see [86] and the references given there) and let \( 1 \leq \phi(k) \leq n_k - n_{k-1} \leq \theta(k) \) for \( k = 2, 3, \ldots \). If each \( f^{(n_k)} \) is univalent in \( D \) and

\[
\limsup_{k \to \infty} \frac{\theta(k) \log \theta(k)}{\phi(k) \log k} = \alpha < 1,
\]

then \( f \) is an entire function of order not greater than \( 1/(1 - \alpha) \).

If however the sequence \( \{n_k\} \) increases very rapidly, say

\[
n_{k+1} \geq n_k \log n_k \log \log n_k,
\]

then \( R \) may not exceed unity. In fact there exists [86] a function \( f \), analytic in \( D \) and an increasing sequence of positive integers \( \{n_k\}_{k=1}^\infty \) such that \( f \) and each \( f^{(n_k)} \) map \( D \) univalently onto convex domains and yet the unit circle is the natural boundary of \( f \).

7. Derivatives with varying radii of univalence. Let \( \rho(f) \) be the largest number with the property that \( f \) is analytic and univalent in an open disc about the origin of radius \( \rho \). We shall write \( \rho(f^{(n)}) = \rho_n \). Suppose now that \( f \) is defined by \( f(z) = \sum_{n=0}^\infty a_n z^n \). Let \( R \) denote the radius of convergence of this series. Then we have [85]

\[
(7.1) \quad \liminf_{n \to \infty} n \rho_n \leq 4R,
\]

and

\[
(7.2) \quad R \log 2 \leq \limsup_{n \to \infty} n \rho_n.
\]

If \( \lvert a_{n-1}/a_n \rvert \) is ultimately a nondecreasing sequence, then

\[
(7.3) \quad R \log 2 \leq \liminf_{n \to \infty} n \rho_n \leq \limsup_{n \to \infty} n \rho_n \leq 4R.
\]

Thus (a) if \( f \) is a transcendental entire function then \( \limsup_{n \to \infty} n \rho_n = \infty \), and (b) if \( \lim_{n \to \infty} n \rho_n = \infty \), then \( f \) is a transcendental entire function. (See also [85, Theorem 5].) The converse of (a) is false. There exists a function \( f \) analytic in the unit disc and in no larger disc \( |z| < R \), where \( R > 1 \), such that \( \limsup n \rho_n = \infty \). The converse of (b) is also false [85].

8. Radii of univalence and entire functions. Let \( f \) be a transcendental entire function of order \( \Lambda \) and lower order \( \lambda \) (see [8, p. 8]). When \( 0 < \Lambda < \infty \), let \( T = \limsup_{r \to \infty} \log M(r)/r^\Lambda \) denote the type and \( t = \)
lim inf_{r \to \infty} \log M(r)/r^\lambda denote the lower type. The following theorems are due to Boas, Pólya and Takenaka respectively.

**THEOREM A [7].** If \( f(z) \) is a transcendental entire function and if

\[
T^* = \limsup_{r \to \infty} \frac{\log M(r)}{r} < \log 2,
\]

then there is a sequence \( \{n_p\}_{p=1}^\infty \) such that \( \rho_{n_p} = \rho(n_p) \geq 1 \) for all \( p \).

Levinson [56] supplied a second proof of this. Boas also pointed out that, if \( T^* = 0 \), then

\[
\limsup_{n \to \infty} \rho_n = \infty.
\]

**THEOREM B [67].** If \( f(z) \) is a transcendental entire function of order \( \Lambda \), then

\[
\liminf_{n \to \infty} \frac{\log \rho_n}{\log n} \leq \frac{1 - \Lambda}{\Lambda} \leq \limsup_{n \to \infty} \frac{\log \rho_n}{\log n}.
\]

**THEOREM C [92].** If \( \{\alpha_n\}_{n=0}^\infty \) is a sequence of complex numbers of modulus not exceeding one and if \( f(z) \) is an entire function of exponential type less than \( \log 2 \), then \( f(z) \) vanishes identically if \( f^{(n)}(\alpha_n) = 0, n = 0, 1, 2, \ldots \).

We give improved versions of these theorems. Let us denote by \( v(r) \) (\( 0 < r < +\infty \)) the central index of the series \( f(z) = \sum_{n=0}^\infty a_n z^n \) for \( |z| = r \). Then

\[
|a_n|r^n \leq |a_{v(r)}|r^{v(r)}, \quad n = 0, 1, 2, \ldots
\]

Let

\[
\limsup_{r \to \infty} \frac{v(r)}{r} = \gamma,
\]

(8.4)

\[
\liminf_{r \to \infty} \frac{v(r)}{r} = \delta.
\]

Then we have [85]

\[
\liminf_{n \to \infty} \frac{\log \max(1, n\rho_n)}{\log n} \leq \frac{1}{\Lambda},
\]

(8.5)

\[
\frac{1 - \lambda}{\lambda} \leq \limsup_{n \to \infty} \frac{\log \rho_n}{\log n},
\]

(8.6)

\[
\frac{\log 2}{\delta} \leq \limsup_{n \to \infty} \rho_n.
\]

(8.7)
and

\[(8.8) \quad \liminf_{n \to \infty} n^{\Lambda - 1} \rho_n^\Lambda \leq \frac{4^\Lambda}{\Lambda T}.\]

Hence if \(\Lambda > 1\), \(\liminf_{n \to \infty} \rho_n = 0\) and if \(\Lambda = 1\), then since \(\delta \leq t^*\)
\((-\liminf_{n \to \infty} \log M(r)/r) \leq T^*\),

\[(8.9) \quad \frac{\log 2}{t^*} \leq \limsup_{n \to \infty} \rho_n; \quad \liminf_{n \to \infty} \rho_n \leq \frac{4}{T^*}.\]

The inequalities (8.5)–(8.6) imply Theorem B and (8.7) implies Theorem A. Theorem C follows immediately from (8.7) since \(\rho(f^{(n)}) \leq r_{n+1}^*\) where \(r_n^*\) denotes the absolute value of the zero \(z_n^*\) of \(f^{(n)}\) which is nearest to the origin. (If \(f^{(k)}\) has no zero then \(r_k^* = \infty\).)

For entire functions defined by gap power series, (8.6) and (8.7) give, in general, better results than Theorems A–C. Let

\[(8.10) \quad f(z) = \sum a_n z^{n_k} \quad (a_{n_k} \neq 0, k = 1, 2, \ldots),\]

be a transcendental entire function and suppose that

\[(8.11) \quad \liminf_{k \to \infty} \log n_k/\log n_{k+1} = \chi < 1.\]

Then \(\lambda \leq \Lambda \chi [93]\) and (8.5) and (8.6) give more information than Theorem B. If we suppose now that \(\Lambda \geq 1\) but \(\Lambda \chi < 1\) then \(\lambda < 1, \delta = 0\) and (8.7) implies that \(\limsup_{n \to \infty} \rho_n = \infty\). Thus Theorems A and C hold for every function \(f\) of any finite order \(\Lambda\) and of the form (8.10) with gaps satisfying the condition (8.11) and \(\Lambda \chi < 1\).

If \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) and \(|a_n/a_{n+1}|\) is ultimately a nondecreasing function of \(n\), tending to \(\infty\), then \(f\) is entire and [85]

\[(8.12) \quad \frac{\log 2}{\gamma} \leq \liminf_{n \to \infty} \rho_n \leq \frac{4}{\gamma},\]

\[(8.13) \quad \frac{\log 2}{\delta} \leq \limsup_{n \to \infty} \rho_n \leq \frac{4}{\delta}.\]

9. **Whittaker constant.** Consider again Theorem A and let \(\alpha\) be the least upper bound of all numbers which can replace \(\log 2\) in Theorem A. Read [71] has shown that \(\alpha \geq 0.7259\). Let \(W\) be the least upper bound of numbers which can replace \(\log 2\) in Theorem C. This number is called the Whittaker constant. It is known that (see [71], [11] and the references given there) \(0.7259 \leq W < 0.7378\) but the exact value is unknown. Recently Buckholtz [11] has shown that \(\alpha = W\).

A simple example of a function \(f\) of order one such that each of \(f, f', f'', \ldots\) has a zero in the closed disc \(|z| \leq 1\) is \(f(z) = \sin(\pi z/4) - \cos(\pi z/4)\).
There exist extremal functions for this problem. In fact Evgrafov (see [11]) has shown that there is an entire function \( f \) of exponential type \( W \) such that each of \( f, f', f'', \ldots \) has a zero in the disc \( |z| \leq 1 \).

Mention must be made here of a related result due to Erdös and Renyi [26]. Let \( f \) be entire and denote by \( x = H(y) \) the inverse function of \( y = \log M(x) \). Then

\[
\liminf_{k \to \infty} \frac{H(k)}{kr_k^2} \leq \frac{e}{\log 2}.
\]

10. **Functions in \( E \).** (i) Consider first a function \( f \) defined by the power series (2.1) and suppose that \( a_n \neq 0, n|(a_n/a_{n-1})| \leq \log 2 \) for \( n = 2, 3, \ldots \). Then \( f \) is entire and it can be shown that \( (f(z) - a_0)/a_1 \in E \).

(ii) We now consider functions with all zeros on a ray. Let \( \Omega \) denote the family of transcendental entire functions \( f \) of the form

\[
f(z) = ze^{\beta z} \prod_{k=1}^{\infty} (1 - z/z_k)
\]

where \( 0 \leq N \leq \infty \) (if \( N = 0 \), the product disappears) and (a) all \( z_k \) have the same argument, (b) \( \beta z_1 \leq 0 \) and (c) \( 1 < |z_1| \leq |z_2| \leq \cdots \). If \( f \in \Omega \) and is univalent in \( D \) then [87]

\[
|\beta| + \sum_{k=1}^{N} \frac{1}{|z_k| - 1} \leq 1.
\]

In fact (10.2) holds if and only if \( f \) is starlike in \( D \) and all its derivatives are close-to-convex there. Further, if \( \{z_k^{(1)}\}_{k=0}^{N} \) are the zeros of \( f' \), then \( f \) and all its derivatives are univalent in \( D \) and map \( D \) onto convex domains if and only if [87]

\[
|\beta| + \sum_{k=0}^{N} \frac{1}{|z_k^{(1)}| - 1} \leq 1.
\]

This result implies that \( E \cap \Omega = S \cap \Omega \) and that \( f \in E \cap \Omega \) if and only if (10.2) holds.

(For the univalence of an entire function of any order see [61].)

(iii) If all zeros of \( f \) do not lie on a ray then some derivative \( f', f'', \ldots \) may have zeros in the unit disc (e.g., \( f(z) = \sin(\pi z/2)/(\pi/2) \)) and then \( f \) will not belong to \( E \). If however \( f \) is of genus zero, and \( f'(0) = 0, f''(0) = 1, \) and the zeros are widely spaced, then \( f \in E \). We shall say that a function \( f \) has “fourly-spaced” zeros if

\[
|z_1| \geq 4, \quad |z_{k+1}| \geq 4^k|z_k|, \quad k \geq 1.
\]
Let
\[ P(z) = \prod_{k=1}^{\infty} (1 - z/z_k), \quad f(z) = z P(z). \]

Then \[ f \in E. \] It is possible to improve the constant 4.

11. **Multivalent functions.** A function \( f \) is said to be \( p \)-valent in \( D \) if it is analytic in \( D \), if the equation
\[
(11.1) \quad f(z) = w
\]
has \( p \) distinct roots in \( D \) for some particular \( w \), and if for each complex \( w \), equation (11.1) does not have more than \( p \) roots in \( D \). The function \( f \) is also said to have valence \( p \) in \( D \). When \( p = 1 \), \( f \) is univalent in \( D \).

Goodman [34] considered the sum \( (f + g)/2 \) and the product \( (fg)^{1/2} \) when \( f \) and \( g \) both belong to \( S \) and showed that there exist two pairs of functions \( f_1, g_1 \) and \( f_2, g_2 \) each function belonging to \( S \) such that the sum \( (f_1 + g_1)/2 \) and, the product \( (f_2(z)g_2(z))^{1/2} = z + \cdots \), both have valence \( \infty \) in \( D \).

We now define areally mean \( p \)-valent (a.m.p.v.) functions. Let \( p \) be a positive number and denote by \( n(w) \) the number of roots of the equation (11.1) in \( D \). If \( f \) is analytic in \( D \) and, for every positive \( R \),
\[
(11.2) \quad \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R n(\rho e^{i\phi}) \rho \, d\rho \, d\phi \leq p,
\]
then \( f \) is said to be a.m.p.v. in \( D \). A condition for \( f \) to be a.m.p.v. is as follows. Let
\[
(11.3) \quad \sum_{n=0}^{\infty} |a_n| = S < |a_0|, \quad \sum_{n=1}^{\infty} n|a_n|^2 = A < \infty.
\]
Then \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is a.m.p.v. in \( D \) for all large \( p \) such that ([39], [68])
\[
(11.4) \quad |a_0| > (A/p)^{1/2} + S.
\]

If \( f \) is a.m.p.v. in \( D \) and is normalized and \( p = 1 \), then \( |a_2| \leq 2 \) [89]. A bound on \( |f| \) is given by the following theorem due to Cartwright, Spencer and Hayman.

**Theorem [38, p. 31].** Suppose that \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) is a.m.p.v. in \( D \). Then
\[
M(r, f) < A(p)\mu_p(1 - r)^{-2p} \quad (0 < r < 1),
\]
where \( \mu_p = \max_{0 \leq r \leq p} |a_r| \) and \( A(p) \leq (p + 2)^{2(p-1)} \exp(p\pi^2 + \frac{1}{2}) \).

This upper bound on the constant \( A(p) \) is due to Jenkins and Oikawa.
In §5(i) we have seen that if $f \in S$ and each $f^{(k)} (k = 1, 2, \ldots)$ is univalent in $D$ then $f$ is a transcendental entire function of exponential type. This result holds under a less restrictive hypothesis. Suppose $f$ is not a polynomial and each $f^{(k)} (k = 0, 1, \ldots)$ is a.m.p.v. in $D$. Then $\|f\|_{\theta} = P$ is an entire function of exponential type not exceeding $A(p)e^{(P + 2)^2(P + 1)}$ where $P = \lfloor p \rfloor$ is integer part of $p$. If each $f^{(n_j)} , j = 1, 2, \ldots$, is a.m.p.v. in $D$

\[(11.5) \lim_{j \to \infty} (n_{j+1} - n_j) = \infty, \quad n_j = O\left(\sum_{k=1}^{j} \log n_k\right),\]

then also $f$ must be entire.

12. Entire functions of bounded index. Let $f(z) = \sum_{n=0}^{\infty} A_n (z - a)^n$ be an entire function. Since the coefficients tend to zero, there exists a smallest integer $N_a \geq 0$ such that $|A_{N_a}| \geq |A_n|$ for all $n$. If the integers $N_a$ are all bounded above then $f$ is said to be of bounded index and the smallest integer $N$, such that for all numbers $a, N_a \leq N$, is called the index of $f$ (cf., [55], [36]). This is equivalent to the definition given in §1. As we pointed out a function of bounded index $N$ is of exponential type not exceeding $N + 1$. This result is sharp [76]. Denote the class of all functions of bounded index by $B$. The functions $e^z$, $\sin z$, $\cos z$ are all in $B$.

The Bessel function $J_k(z)$ of integer order $k$ is of index $N$ such that $k \leq N \leq 2k - 1$ ([52]; see also [60]). Any entire function $f$ satisfying a linear differential equation [77]

\[(12.1) \quad P_0(z) \frac{d^n f}{dz^n} + P_1(z) \frac{d^{n-1} f}{dz^{n-1}} + \cdots + P_n(z)f = Q(z),\]

where $P_j (j = 0, 1, \ldots, n)$ and $Q$ are polynomials and $\deg P_j \leq \deg P_0$ is in class $B$.

Functions with zeros of arbitrarily large multiplicity are obviously of unbounded index. But there are functions [79] of unbounded index and having simple zeros.

The asymptotic properties of $\log M(r, f)$ do not help to prove the boundedness (or the unboundedness) of the index, except that if $T* = \infty$ then $f \in CB$ (the class of entire functions of unbounded index). In fact if $F$ is any transcendental entire function then there are two entire functions $g \in CB$ [70] and $f \in CE$ (the class of entire functions not belonging to $E$) such that

$$\log M(r, g) \sim \log M(r, F) \sim \log M(r, f).$$

For $f$ we simply take $f(z) = F(z) - F''(0)z^2/2!$.

We mentioned in §11 that there exist functions $f$ and $g$ in $S$ such that $(f + g)/2$ is not in $S$. Pugh [69] showed that the sum of two functions each in $B$, need not be in $B$.  

[45].
The class $B$ is not closed under differentiation. There exists [80] an entire function $F$ in $B$ such that the derivative $F'$ is in $CB$. If the derivative $f'$ is of bounded index $N_f'$, $f$ is also of bounded index $N_f$ and $N_f \leq N_f' + 1$ [80].

The functions $P$ and $f$ defined by (10.4) and (10.5) are both in $B$. (Cf. [70]. The constant 5 in [70] has been improved to 4 by Mrs. Amy King in her Ph.D dissertation.) In fact, we have, for all $z$,

$$\max\{|P(z)|, |P'(z)| \geq |P^{(n)}(z)|, \quad n = 2, 3, \ldots.$$

Furthermore each $P^{(k)}$, $k = 0, 1, 2, \ldots$, is of index 1.

Consider now functions with real zeros $a_n$. Suppose $a_1 > 0, a_{n+1} - a_n \geq b_n$ ($n \geq 1$) where the sequence $\{b_n\}_1^\infty$ is positive and nondecreasing and $\sum_1^\infty 1/nb_n < \infty$. Then [82],

$$f(z) = e^{az+\beta} \prod_1^\infty \left(1 - \frac{z}{an}\right),$$

where $a$ and $\beta$ are any complex numbers, is in $B$. If in (12.2) we assume that $a_1 > 0, a_{n+1}/a_n \geq \gamma > 1$, then each $f^{(k)}$, $k = 0, 1, \ldots$, is in $B$ [54].

We can consider entire functions $f$ satisfying conditions similar to (1.1) and obtain the conclusion that $f$ must be of exponential type [37], [83].

(a) Let $p \geq 1$ and

$$I(a, r) = \left\{ \int_0^{2\pi} |f^{(l)}(re^{i\theta})|^p \, d\theta \right\}^{1/p}. $$

Let $c$ be a positive constant. Suppose that there exists a positive integer $N$ (independent of $z$) such that for $k = 0, 1, 2, \ldots, N$, the following inequality

$$\sum_{j=0}^N \frac{I(k+j, r)}{j!} \geq c \sum_{j=N+1}^\infty \frac{I(k+j, r)}{j!}$$

holds for all $z$ with $|z| = r$ sufficiently large. Then $f$ is of exponential type and

$$T^* \leq 1 + 2 \log(1 + 1/c) + \log(2N)!.$$

(b) Let $c$ be a positive constant. Suppose that there exist two non-negative integers $k$ and $N$ (independent of $z$) such that $f$ satisfies one of the following, for all $z$ with $|z|$ sufficiently large:

(i) $$\sum_{j=0}^N \frac{|f^{(k+j)}(z)|}{j!} \geq c \sum_{j=N+1}^\infty \frac{|f^{(k+j)}(z)|}{j!},$$

(ii) $$\sum_{j=0}^N \frac{M(r, f^{(k+j)})}{j!} \geq c \sum_{j=N+1}^\infty \frac{M(r, f^{(k+j)})}{j!},$$
then $f$ is of exponential type and

$$T^* \leq \max \left\{ N, \min_{1 \leq j \leq N} \left( \frac{(N + j)!(N + 1)!}{(N!)^j} \right)^{1/j}, \left( \frac{(2N + 1)1/(N + 1)!}{(N!)^j} \right)^{1/(N+1)} \right\}.$$

13. **The space of entire functions.** Following Iyer [43] we define a metric on the space of all entire functions $f$. (This space includes all polynomials and constant zero.) Let $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g(z) = \sum_{n=0}^\infty b_n z^n \in \Gamma$ and define

$$d(f, g) = \sup \{|a_0 - b_0|, |a_n - b_n|^{1/n}: n = 1, 2, \ldots\}.$$

Then $d$ is a metric and $(\Gamma, d)$ is a complete metric space [43]. Let

$$B_n = \{ f \in (\Gamma, d) | f \text{ is of index not exceeding } n \}.$$

We consider $B = \bigcup_{n=0}^\infty B_n$ as a subspace of $(\Gamma, d)$. It can be shown that [25] $B_n$ is nowhere dense in $B$ and thus $B$ is of the first category.

14. **Some applications to summability methods.** Let $f$ be entire and \{\(z_i\)\}_{i=0}^\infty a sequence of complex numbers. We define the matrix transformation $A(f, z_i) = (a_{n,k})$ by

$$f(z) = \sum_{k=0}^\infty a_{n,k}(z - z_n)^k \text{ for } n = 0, 1, \ldots.$$

We now state some recent results of Fricke and Powell.

I [28]. If $f \in B$ then $A(f, z_i) = (a_{n,k})$ is not regular for any sequence \{\(z_i\)\}_{i=0}^\infty. (A transformation $A = (a_{n,k})$ is regular if it transforms every convergent sequence into a sequence converging to the same limit. See [41, p. 43].)

Define a sequence \{\(a_n\)\}_{n=0}^\infty to be entire if $f(z) = \sum_{n=0}^\infty a_n z^n$ is an entire function. An entire sequence \{\(a_n\)\}_{n=0}^\infty is said to be a sequence of bounded index if $f(z) = \sum_{n=0}^\infty a_n z^n \in B$. We denote by $e$ the set of all entire sequences and by $B$ the set of all entire sequences of bounded index. An infinite matrix $A = (a_{n,k})$ of complex numbers which transforms $e$ into $e$ is said to be an $\varepsilon$-$\varepsilon$ method (entire method).

II [27]. A matrix $A = (a_{n,k})$ is an $\varepsilon$-$\varepsilon$ method if and only if for each integer $q > 0$, there exists an integer $p > 0$ and a constant $M > 0$ such that

$$|a_{n,k}q^n| \leq Mp^k \text{ for all } n, k = 0, 1, \ldots.$$

Let $A'(f, z_i) = (b_{n,k})$ denote the transpose of $A(f, z_i) = (a_{n,k})$, that is, $b_{n,k} = a_{k,n}$.

III [28]. If $f \in B$ then for any sequence \{\(z_i\)\}_{i=0}^\infty, $A'(f, z_i) = (b_{n,k})$ is an $\varepsilon$-$\varepsilon$ method if and only if for each integer $n > 0$ there exist an integer $p > 0$ and a constant $M > 0$ such that

$$|f^{(n)}(z_k)| \leq p^k M \text{ for } k = 0, 1, \ldots.$$
The condition that \( f \in B \) is essential in III.

We now define the \( l-l \) method. Let \( s \) be the set of all sequences of complex numbers. Let

\[
\ell = \left\{ x = \{x_n\}_{n=0}^\infty \in s \mid \sum_{n=0}^\infty |x_n| < \infty \right\}.
\]

A matrix \( A = (a_{n,k}) \) that maps \( \ell \) into itself is said to be an \( l-l \) method. Knopp and Lorentz [49] proved that a matrix \( A = (a_{n,k}) \) is an \( l-l \) method if and only if there exists a constant \( M > 0 \) such that

\[
\sum_{n=0}^\infty |a_{n,k}| \leq M \quad \text{for } k = 0, 1, \ldots.
\]

IV [28]. Let \( f \in B \) and \( \{z_i\}_{i=0}^\infty \) be a sequence of complex numbers. If either \( A(f, z_i) = (a_{n,k}) \) or \( A'(f, z_i) = (b_{n,k}) \) is an \( l-l \) method then \( A'(f, z_i) \) is an \( e-e \) method.

Finally we give a matrix which transforms \( B \) into \( B \).

Let the Taylor matrix \( T(\xi) = (a_{n,k}) \) be defined by

\[
a_{n,k} = \begin{cases} \frac{k}{n} (1 - \xi)^{n+1} \xi^{k-n}, & \text{if } k \geq n, \\ 0, & \text{otherwise}, \end{cases}
\]

where \( \xi \) is a complex number.

V [28]. The Taylor matrix \( T(\xi) = (a_{n,k}) \) transforms \( B \) into \( B \) for any complex number \( \xi \).

15. Conjectures and open problems. We now list some problems and conjectures connected with two classes \( E \) and \( B \).

**Conjecture 1.** If \( \phi \) is any transcendental entire function such that

\[
\lim_{r \to \infty} \sup \frac{\log M(r, \phi)}{r} \leq \pi,
\]

there exists an entire function \( f \in E \) such that \( \log M(r, \phi) \sim \log M(r, f) \) \((r \to \infty)\).

**Conjecture 2.** If \( \phi \) is any entire function of exponential type, there exists an entire function \( f \in B \) such that \( \log M(r, \phi) \sim \log M(r, f) \) \((r \to \infty)\).

For some theorems of this type, but not connected with \( E \) or \( B \), see [22], [19], [20].

**Conjecture 3.** \( W = 2/e \).

**Conjecture 4.** If \( \sum_{p=1}^\infty 1/n_p = \infty \) and \( \rho(f^{(n_p)}) \geq 1 \) for \( p = 1, 2, \ldots \), then \( f \) is entire.

In the following problems 1–4, \( f \in E \).
1. What is the smallest zero that $f$ can have? (Exclude $z = 0$.)
2. What is the largest circle center origin covered by $f(D)$?
3. Find bounds on $|a_2^3 - a_3|$.
4. Find $\alpha = \sup\{|a_2| \mid f \in E\}$.
5. Find $\alpha_K = \sup\{|a_2| \mid f \in T(K)\}$.
6. Let $f$ be entire and satisfy a differential equation of the form (12.1). Assume $P_j (j = 0, 1, \ldots, n)$ and $Q$ are polynomials and $\deg P_j \leq \deg P_0$. Then $f$ is of bounded index. Find an estimate for the index.

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