The Grunsky inequalities characterize the analytic functions that are univalent. Theorem 1 gives a new set of inequalities which appear to be the result of exponentiating the Grunsky inequalities for functions on the unit disc.

**Theorem 1.** If \( f(z) = z + a_2 z^2 + \cdots \) is a one-to-one, analytic function on \( \{ z : |z| < 1 \} \), then

\[
\frac{1}{n} \left| \sum_{v=1}^{n} \alpha_v \frac{f(z_v) - f(z_\mu)}{z_v - z_\mu} \right| \leq \frac{1}{n} \left| \sum_{v=1}^{n} \alpha_v \frac{1}{1 - z_v z_\mu} \right|
\]

for all \( z_v \) in the unit disc and all complex numbers \( \alpha_v \) for \( n = 1, 2, \ldots \). For \( z_v = z_\mu \) replace \( (z_v - z_\mu)/(f(z_v) - f(z_\mu)) \) by \( 1/f'(z_v) \).

This theorem can be proved by an extension by Goluzin's method \[2\] of using Löwner's differential equation \[4\] to prove the Grunsky inequalities. Using (1), it is easy to find the bounds on the coefficients of the inverse function \( f^{-1}(w) \) for all functions \( f \) as described in Theorem 1. (This problem was first solved by Löwner \[4\].)

By the same method, the following theorem can be proved.

**Theorem 2.** If \( f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \) is a one-to-one, analytic function on \( \{ z : |z| < 1 \} \), then

\[
\frac{1}{n} \left| \sum_{v=1}^{n} \alpha_v \frac{f(z_v) - f(z_\mu)}{z_v - z_\mu} \right| \leq \frac{1}{n} \left| \sum_{v=1}^{n} \alpha_v \frac{1}{1 - z_v z_\mu} \right|
\]

and

\[
\frac{1}{n} \left| \sum_{v=1}^{n} \alpha_v \frac{f(z_v) - f(z_\mu)}{z_v - z_\mu} \right|^2 \geq \frac{1}{n} \left| \sum_{v=1}^{n} \alpha_v \frac{1}{z_v} \right|^2 \]

for all \( z_v \) in the unit disc, for all complex numbers \( \alpha_v \) and \( n = 1, 2, \ldots \). For \( z_v = z_\mu \) replace \( (f(z_v) - f(z_\mu))/(z_v - z_\mu) \) by \( f'(z_v) \).

From (2) it follows that if the coefficients of \( f \) are all real, then \( a_1 + a_3 + \cdots + a_{2n-1} \geq a_n^2 \) and consequently \( |a_n| \leq n \) for \( n = 1, 2, \ldots \). (That the

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Bieberbach conjecture holds for functions with real coefficients was first proved by Dieudonné [1] and Rogosinski [6].

From (3) it follows that

\[ \sum_{k=1}^{n} k|a_k|^2 + \sum_{k=n+1}^{2n-1} (2n-k)|a_k|^2 \geq |a_n|^4 \]

and consequently \( |a_n| \leq (7/6)^{1/2} n \) for \( n = 1, 2, \ldots \). The constant \( (7/6)^{1/2} \) is not the smallest that follows from inequality (3), but this estimate already compares favorably with the best previous result \( |a_n| \leq (1.243)n \) obtained by Milin [5].

From (3) also follows a more general inequality than (4) which implies that \( \limsup_{n \to \infty} |a_n|/n < 1 \), except in case \( f(z) = z/(1 - e^{i\theta}z)^2 \). (This theorem was first proved by Hayman [3].)

REFERENCES


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